

# Arithmetic harmonic analysis on character and quiver varieties II

Tamás Hausel

*University of Oxford*

hausel@maths.ox.ac.uk

Emmanuel Letellier

*Université de Caen*

letellier.emmanuel@math.unicaen.fr

Fernando Rodriguez-Villegas

*University of Texas at Austin*

villegas@math.utexas.edu

with an appendix by Gergely Harcos

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## Abstract

We study connections between the topology of generic character varieties of fundamental groups of punctured Riemann surfaces, Macdonald polynomials, quiver representations, Hilbert schemes on  $\mathbb{C}^\times \times \mathbb{C}^\times$ , modular forms and multiplicities in tensor products of irreducible characters of finite general linear groups.

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# 1 Introduction

## 1.1 Character varieties

Given a non-negative integer  $g$  and a  $k$ -tuple  $\mu = (\mu^1, \mu^2, \dots, \mu^k)$  of partitions of  $n$ , we define the generic character variety  $\mathcal{M}_\mu$  of type  $\mu$  as follows (see [10] for more details). Choose a *generic* tuple  $(C_1, \dots, C_k)$  of semisimple conjugacy classes of  $\mathrm{GL}_n(\mathbb{C})$  such that for each  $i = 1, 2, \dots, k$  the multiplicities of the eigenvalues of  $C_i$  are given by the parts of  $\mu^i$ .

Define  $\mathcal{Z}_\mu$  as

$$\mathcal{Z}_\mu := \left\{ (a_1, b_1, \dots, a_g, b_g, x_1, \dots, x_k) \in (\mathrm{GL}_n)^{2g} \times C_1 \times \dots \times C_k \left| \prod_{j=1}^g (a_j, b_j) \prod_{i=1}^k x_i = 1 \right. \right\},$$

where  $(a, b) = aba^{-1}b^{-1}$ . The group  $\mathrm{GL}_n$  acts diagonally by conjugation on  $\mathcal{Z}_\mu$  and we define  $\mathcal{M}_\mu$  as the affine GIT quotient

$$\mathcal{M}_\mu := \mathcal{Z}_\mu // \mathrm{GL}_n := \mathrm{Spec} \left( \mathbb{C}[\mathcal{Z}_\mu]^{\mathrm{GL}_n} \right).$$

We prove in [10] that, if non-empty,  $\mathcal{M}_\mu$  is nonsingular of pure dimension

$$d_\mu := n^2(2g - 2 + k) - \sum_{i,j} (\mu_j^i)^2 + 2.$$

We also defined an *a priori* rational function  $\mathbb{H}_\mu(z, w) \in \mathbb{Q}(z, w)$  in terms of Macdonald symmetric functions (see § 2.1.4 for a precise definition) and we conjecture that the compactly supported mixed Hodge numbers  $\{h_c^{i,j,k}(\mathcal{M}_\mu)\}_{i,j,k}$  satisfies  $h_c^{i,j,k}(\mathcal{M}_\mu) = 0$  unless  $i = j$  and

$$H_c(\mathcal{M}_\mu; q, t) \stackrel{?}{=} (t\sqrt{q})^{d_\mu} \mathbb{H}_\mu \left( -t\sqrt{q}, \frac{1}{\sqrt{q}} \right), \quad (1.1.1)$$

where  $H_c(\mathcal{M}_\mu; q, t) := \sum_{i,j} h_c^{i,i,j}(\mathcal{M}_\mu) q^i t^j$  is the compactly supported mixed Hodge polynomial.

In particular,  $\mathbb{H}_\mu(-z, w)$  should actually be a polynomial with non-negative integer coefficients of degree  $d_\mu$  in each variable.

In [10] we prove that (1.1.1) is true under the specialization  $(q, t) \mapsto (q, -1)$ , namely,

$$E(\mathcal{M}_\mu; q) := H_c(\mathcal{M}_\mu; q, -1) = q^{\frac{1}{2}d_\mu} \mathbb{H}_\mu \left( \sqrt{q}, \frac{1}{\sqrt{q}} \right). \quad (1.1.2)$$

This formula is obtained by counting points of  $\mathcal{M}_\mu$  over finite fields (after choosing a spreading out of  $\mathcal{M}_\mu$  over a finitely generated subalgebra of  $\mathbb{C}$ ). We compute  $\mathcal{M}_\mu(\mathbb{F}_q)$  using a formula involving the values of the irreducible characters of  $\mathrm{GL}_n(\mathbb{F}_q)$  (a formula that goes back to Frobenius [5]). The calculation shows that  $\mathcal{M}_\mu$  is *polynomial count*; i.e., there exists a polynomial  $P \in \mathbb{C}[T]$  such that for any finite field  $\mathbb{F}_q$  of sufficiently large characteristic,  $\#\mathcal{M}_\mu(\mathbb{F}_q) = P(q)$ . Then by a theorem of Katz [10, Appendix]  $E(\mathcal{M}_\mu; q) = P(q)$ .

Recall also that the  $E(\mathcal{M}_\mu; q)$  satisfies the following identity

$$E(\mathcal{M}_\mu; q) = q^{d_\mu} E(\mathcal{M}_\mu; q^{-1}). \quad (1.1.3)$$

In this paper we use Formula (1.1.2) to prove the following theorem.

**Theorem 1.1.1.** *If non-empty, the character variety  $\mathcal{M}_\mu$  is connected.*

The proof of the theorem reduces to proving that the coefficient of the lowest power of  $q$  in  $\mathbb{H}_\mu(\sqrt{q}, 1/\sqrt{q})$ , namely  $q^{-d_\mu/2}$ , equals 1. This turns out to require a rather delicate argument, by far the most technical of the paper, that uses the inequality of § 6 in a crucial way.

## 1.2 Relations to Hilbert schemes on $\mathbb{C}^\times \times \mathbb{C}^\times$ and modular forms

Here we assume that  $g = k = 1$ . Put  $X = \mathbb{C}^\times \times \mathbb{C}^\times$  and denote by  $X^{[n]}$  the Hilbert scheme of  $n$  points in  $X$ . Define  $\mathbb{H}^{[n]}(z, w) \in \mathbb{Q}(z, w)$  by

$$\sum_{n \geq 0} \mathbb{H}^{[n]}(z, w) T^n := \prod_{n \geq 1} \frac{(1 - zwT^n)^2}{(1 - z^2T^n)(1 - w^2T^n)}, \quad (1.2.1)$$

with the convention that  $\mathbb{H}^{[0]}(z, w) := 1$ . It is known by work of Göttsche and Soergel [9] that the mixed Hodge polynomial  $H_c(X^{[n]}; q, t)$  is given by

$$H_c(X^{[n]}; q, t) = (qt^2)^n \mathbb{H}^{[n]} \left( -t\sqrt{q}, \frac{1}{\sqrt{q}} \right).$$

**Conjecture 1.2.1.** *We have*

$$\mathbb{H}^{[n]}(z, w) = \mathbb{H}_{(n-1,1)}(z, w).$$

This together with the conjectural formula (1.1.1) implies that the Hilbert scheme  $X^{[n]}$  and the character variety  $\mathcal{M}_{(n-1,1)}$  should have the same mixed Hodge polynomial. Although this is believed to be true (in the analogous additive case this is well-known; see Theorem 4.1.1) there is no complete proof in the literature. (The result follows from known facts modulo some missing arguments in the non-Abelian Hodge theory for punctured Riemann surfaces; see the comment after Conjecture 4.2.1.) We prove the following results which give evidence for Conjecture 1.2.1.

**Theorem 1.2.2.** *We have*

$$\begin{aligned} \mathbb{H}^{[n]}(0, w) &= \mathbb{H}_{(n-1,1)}(0, w), \\ \mathbb{H}^{[n]}(w^{-1}, w) &= \mathbb{H}_{(n-1,1)}(w^{-1}, w). \end{aligned}$$

The second identity means that the  $E$ -polynomials of  $X^{[n]}$  and  $\mathcal{M}_{(n-1,1)}$  agree. As a consequence of Theorem 1.2.2 we have the following relation between character varieties and quasi-modular forms.

**Corollary 1.2.3.** *We have*

$$1 + \sum_{n \geq 1} \mathbb{H}_{(n-1,1)}(e^{u/2}, e^{-u/2}) T^n = \frac{1}{u} (e^{u/2} - e^{-u/2}) \exp \left( 2 \sum_{k \geq 2} G_k(T) \frac{u^k}{k!} \right),$$

where

$$G_k(T) = \frac{-B_k}{2k} + \sum_{n \geq 1} \sum_{d|n} d^{k-1} T^n$$

(with  $B_k$  is the  $k$ -th Bernoulli number) is the classical Eisenstein series for  $SL_2(\mathbb{Z})$ .

In particular, the coefficient of any power of  $u$  in the left hand side is in the ring of quasi-modular forms, generated by the  $G_k$ ,  $k \geq 2$ , over  $\mathbb{Q}$ .

Relation between Hilbert schemes and modular forms was first investigated by Göttsche [8].

### 1.3 Quiver representations

For a partition  $\mu = \mu_1 \geq \dots \geq \mu_r > 0$  of  $n$  we denote by  $l(\mu) = r$  its length. Given a non-negative integer  $g$  and a  $k$ -tuple  $\boldsymbol{\mu} = (\mu^1, \mu^2, \dots, \mu^k)$  of partitions of  $n$  we define a *comet-shaped* quiver  $\Gamma_{\boldsymbol{\mu}}$  with  $k$  legs of length  $s_1, s_2, \dots, s_k$  (where  $s_i = l(\mu^i) - 1$ ) and with  $g$  loops at the central vertex (see picture in §3.2). The multi-partition  $\boldsymbol{\mu}$  defines also a dimension vector  $\mathbf{v}_{\boldsymbol{\mu}}$  of  $\Gamma_{\boldsymbol{\mu}}$  whose coordinates on the  $i$ -th leg are  $(n, n - \mu_1^i, n - \mu_1^i - \mu_2^i, \dots, n - \sum_{r=1}^{s_i} \mu_r^i)$ .

By a theorem of Kac [16] there exists a monic polynomial  $A_{\boldsymbol{\mu}}(T) \in \mathbb{Z}[T]$  of degree  $d_{\boldsymbol{\mu}}/2$  such that the number of absolutely indecomposable representations over  $\mathbb{F}_q$  (up to isomorphism) of  $\Gamma_{\boldsymbol{\mu}}$  of dimension  $\mathbf{v}_{\boldsymbol{\mu}}$  equals  $A_{\boldsymbol{\mu}}(q)$ .

Let us state our first main result.

**Theorem 1.3.1.** *We have*

$$A_{\boldsymbol{\mu}}(q) = \mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q}). \quad (1.3.1)$$

If we assume that  $\mathbf{v}_{\boldsymbol{\mu}}$  is indivisible, i.e., the gcd of all the parts of the partitions  $\mu^1, \dots, \mu^k$  equals 1, then, as mentioned in [10, Remark 1.4.3], the formula can be proved using the results of Crawley-Boevey and van den Bergh [1] together with the results in [10]. More precisely the results of Crawley-Boevey and van den Bergh say that  $A_{\boldsymbol{\mu}}(q)$  equals (up to some power of  $q$ ) the compactly supported Poincaré polynomial of some quiver variety  $Q_{\boldsymbol{\mu}}$  (which exists only if  $\mathbf{v}_{\boldsymbol{\mu}}$  is indivisible). In [10] we show that the Poincaré polynomial of  $Q_{\boldsymbol{\mu}}$  agrees with  $\mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q})$  up to the same power of  $q$ , hence the formula (1.3.1).

The proof of Formula (1.3.1) we give in this paper is completely combinatorial (and works also in the divisible case). It is based on Hua's formula [14] for the number of absolutely indecomposable representations of quivers over finite fields.

The conjectural formula (1.1.1) together with Formula (1.3.1) implies the following conjecture.

**Conjecture 1.3.2.** *We have*

$$A_{\boldsymbol{\mu}}(q) = q^{-\frac{d_{\boldsymbol{\mu}}}{2}} PH_c(\mathcal{M}_{\boldsymbol{\mu}}; q),$$

where  $PH_c(\mathcal{M}_{\boldsymbol{\mu}}; q) := \sum_i h_c^{i, i+2i}(\mathcal{M}_{\boldsymbol{\mu}}) q^i$  is the pure part of  $H_c(\mathcal{M}_{\boldsymbol{\mu}}; q, t)$ .

Conjecture 1.3.2 implies Kac's conjecture [16] for comet shaped quivers, namely,  $A_{\boldsymbol{\mu}}(q)$  is a polynomial in  $q$  with non-negative coefficients (see §3.1 for more details).

### 1.4 Characters of general linear groups over finite fields

Given two irreducible complex characters  $X_1, X_2$  of  $\mathrm{GL}_n(\mathbb{F}_q)$  it is a natural and difficult question to understand the decomposition of the tensor product  $X_1 \otimes X_2$  as a sum of irreducible characters. Note that the character table of  $\mathrm{GL}_n(\mathbb{F}_q)$  is known (Green, 1955) and so we can compute in theory the multiplicity  $\langle X_1 \otimes X_2, X \rangle$  of any irreducible character  $X$  of  $\mathrm{GL}_n(\mathbb{F}_q)$  in  $X_1 \otimes X_2$  using the scalar product formula

$$\langle X_1 \otimes X_2, X \rangle = \frac{1}{|\mathrm{GL}_n(\mathbb{F}_q)|} \sum_{g \in \mathrm{GL}_n(\mathbb{F}_q)} X_1(g) X_2(g) \overline{X(g)}. \quad (1.4.1)$$

However it is very difficult to extract any interesting information from this formula. In his thesis Mattig uses this formula to compute (with the help of a computer) the multiplicities  $\langle X_1 \otimes X_2, X \rangle$  when  $X_1, X_2, X$  are *unipotent characters* and when  $n \leq 8$  (see [13]), and he noticed that  $\langle X_1 \otimes X_2, X \rangle$  is a polynomial in  $q$  with positive integer coefficients.

In [10] we define the notion of *generic* tuple  $(X_1, \dots, X_k)$  of irreducible characters of  $\mathrm{GL}_n(\mathbb{F}_q)$ . We also consider the character  $\Lambda : \mathrm{GL}_n(\mathbb{F}_q) \rightarrow \mathbb{C}$ ,  $x \mapsto q^{g \cdot \dim C_{\mathrm{GL}_n}(x)}$  where  $C_{\mathrm{GL}_n}(x)$  denotes the centralizer of  $x$  in  $\mathrm{GL}_n(\mathbb{F}_q)$  and where  $g$  is a non-negative integer. If  $g = 1$ , this is the character of the conjugation action of  $\mathrm{GL}_n(\mathbb{F}_q)$  on the group algebra  $\mathbb{C}[\mathrm{gl}_n(\mathbb{F}_q)]$ .

If  $\mu = (\mu_1, \mu_2, \dots, \mu_r)$  is a partition of  $n$ , an irreducible character of  $\mathrm{GL}_n(\mathbb{F}_q)$  is said to be of type  $\mu$  if it is of the form  $R_{L_\mu}^{GL_n}(\alpha)$  where  $L_\mu = \mathrm{GL}_{\mu_1} \times \mathrm{GL}_{\mu_2} \times \dots \times \mathrm{GL}_{\mu_r}$  and where  $\alpha$  is a *regular* linear character of  $L_\mu(\mathbb{F}_q)$ , see §3.4 for definitions. Characters of this form are called *semisimple split*.

In [10] we prove that for a generic tuple  $(X_1, \dots, X_k)$  of semisimple split irreducible characters of  $\mathrm{GL}_n(\mathbb{F}_q)$  of type  $\mu$ , we have

$$\langle \Lambda \otimes X_1 \otimes \dots \otimes X_k, 1 \rangle = \mathbb{H}_\mu(0, \sqrt{q}). \quad (1.4.2)$$

Note that in particular this implies that the left hand side only depends on the combinatorial type  $\mu$  not on the specific choice of characters.

Together with Formula (1.3.1) we deduce the following formula.

**Theorem 1.4.1.** *We have*

$$\langle \Lambda \otimes X_1 \otimes \dots \otimes X_k, 1 \rangle = A_\mu(q).$$

Using Kac's results on quiver representations (see §3.1) the above theorem has the following consequence.

**Corollary 1.4.2.** *Let  $\Phi(\Gamma_\mu)$  denote the root system associated with  $\Gamma_\mu$  and let  $(X_1, \dots, X_k)$  be a generic  $k$ -tuple of irreducible characters of  $\mathrm{GL}_n(\mathbb{F}_q)$  of type  $\mu$ .*

*We have  $\langle \Lambda \otimes X_1 \otimes \dots \otimes X_k, 1 \rangle \neq 0$  if and only if  $\mathbf{v}_\mu \in \Phi(\Gamma_\mu)$ . Moreover  $\langle \Lambda \otimes X_1 \otimes \dots \otimes X_k, 1 \rangle = 1$  if and only if  $\mathbf{v}_\mu$  is a real root.*

In [22] the second author proves that Corollary 1.4.2 extends to any type of generic tuples of irreducible characters of  $\mathrm{GL}_n(\mathbb{F}_q)$  (not necessarily semisimple split).

Recall that there is a natural parametrization,  $\mu \mapsto \mathcal{U}_\mu$ , of the unipotent characters of  $\mathrm{GL}_n(\mathbb{F}_q)$  by partitions of  $n$ , fixed by requiring that  $\mathcal{U}_{(1^n)}$  is trivial. Using again quiver representations, we also prove in §3.4 the following result on multiplicities for unipotent characters (confirming partly Mattig's observation for  $n \leq 8$ ).

**Proposition 1.4.3.** *For a  $k$ -tuple of partitions  $\mu = (\mu^1, \dots, \mu^k)$  of  $n$  there exists a polynomial  $U_\mu \in \mathbb{Z}[T]$  such that  $\langle \Lambda \otimes \mathcal{U}_{\mu^1} \otimes \dots \otimes \mathcal{U}_{\mu^k}, 1 \rangle = U_\mu(q)$ .*

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## 2 Preliminaries

We denote by  $\mathbb{F}$  an algebraic closure of a finite field  $\mathbb{F}_q$ .

### 2.1 Symmetric functions

#### 2.1.1 Partitions, Macdonald polynomials, Green polynomials

We denote by  $\mathcal{P}$  the set of all partitions including the unique partition 0 of 0, by  $\mathcal{P}^*$  the set of non-zero partitions and by  $\mathcal{P}_n$  be the set of partitions of  $n$ . Partitions  $\lambda$  are denoted by  $\lambda = (\lambda_1, \lambda_2, \dots)$ , where  $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$ . We will also sometimes write a partition as  $(1^{m_1}, 2^{m_2}, \dots, n^{m_n})$  where  $m_i$  denotes the multiplicity of  $i$  in  $\lambda$ . The *size* of  $\lambda$  is  $|\lambda| := \sum_i \lambda_i$ ; the *length*  $l(\lambda)$  of  $\lambda$  is the maximum  $i$  with  $\lambda_i > 0$ . For two partitions  $\lambda$  and  $\mu$ , we define  $\langle \lambda, \mu \rangle$  as  $\sum_i \lambda'_i \mu'_i$  where  $\lambda'$  denotes the dual partition of  $\lambda$ . We put

$n(\lambda) = \sum_{i>0} (i-1)\lambda_i$ . Then  $\langle \lambda, \lambda \rangle = 2n(\lambda) + |\lambda|$ . For two partitions  $\lambda = (1^{n_1}, 2^{n_2}, \dots)$  and  $\mu = (1^{m_1}, 2^{m_2}, \dots)$ , we denote by  $\lambda \cup \mu$  the partition  $(1^{n_1+m_1}, 2^{n_2+m_2}, \dots)$ . For a non-negative integer  $d$  and a partition  $\lambda$ , we denote by  $d \cdot \lambda$  the partition  $(d\lambda_1, d\lambda_2, \dots)$ . The *dominance ordering* for partitions is defined as follows:  $\mu \leq \lambda$  if and only if  $\mu_1 + \dots + \mu_j \leq \lambda_1 + \dots + \lambda_j$  for all  $j \geq 1$ .

Let  $\mathbf{x} = \{x_1, x_2, \dots\}$  be an infinite set of variables and  $\Lambda(\mathbf{x})$  the corresponding ring of symmetric functions. As usual we will denote by  $s_\lambda(\mathbf{x})$ ,  $h_\lambda(\mathbf{x})$ ,  $p_\lambda(\mathbf{x})$ , and  $m_\lambda(\mathbf{x})$ , the Schur symmetric functions, the complete symmetric functions, the power symmetric functions and the monomial symmetric functions.

We will deal with elements of the ring  $\Lambda(\mathbf{x}) \otimes_{\mathbb{Z}} \mathbb{Q}(z, w)$  and their images under two specializations: their *pure part*,  $z = 0, w = \sqrt{q}$  and their *Euler specialization*,  $z = \sqrt{q}, w = 1/\sqrt{q}$ .

For a partition  $\lambda$ , let  $\tilde{H}_\lambda(\mathbf{x}; q, t) \in \Lambda(\mathbf{x}) \otimes_{\mathbb{Z}} \mathbb{Q}(q, t)$  be the *Macdonald symmetric function* defined in Garsia and Haiman [7, I.11]. We collect in this section some basic properties of these functions that we will need.

We have the duality

$$\tilde{H}_\lambda(\mathbf{x}; q, t) = \tilde{H}_{\lambda'}(\mathbf{x}; t, q) \quad (2.1.1)$$

see [7, Corollary 3.2]. We define the (transformed) *Hall-Littlewood symmetric function* as

$$\tilde{H}_\lambda(\mathbf{x}; q) := \tilde{H}_\lambda(\mathbf{x}; 0, q). \quad (2.1.2)$$

In the notation just introduced then  $\tilde{H}_\lambda(\mathbf{x}; q)$  is the pure part of  $\tilde{H}_\lambda(\mathbf{x}; z^2, w^2)$ .

Under the Euler specialization of  $\tilde{H}_\lambda(\mathbf{x}; z^2, w^2)$  we have [10, Lemma 2.3.4]

$$\tilde{H}_\lambda(\mathbf{x}; q, q^{-1}) = q^{-n(\lambda)} H_\lambda(q) s_\lambda(\mathbf{xy}), \quad (2.1.3)$$

where  $y_i = q^{i-1}$  and  $H_\lambda(q) := \prod_{s \in \lambda} (1 - q^{h(s)})$  is the *hook polynomial* [24, I, 3, example 2].

Define the  $(q, t)$ -*Kostka polynomials*  $\tilde{K}_{\nu\lambda}(q, t)$  by

$$\tilde{H}_\lambda(\mathbf{x}; q, t) = \sum_{\nu} \tilde{K}_{\nu\lambda}(q, t) s_\nu(\mathbf{x}). \quad (2.1.4)$$

These are  $(q, t)$  generalizations of the  $\tilde{K}_{\nu\lambda}(q)$  Kostka-Foulkes polynomial in Macdonald [24, III, (7.11)], which are obtained as  $q^{n(\lambda)} K_{\nu\lambda}(q^{-1}) = \tilde{K}_{\nu\lambda}(q) = \tilde{K}_{\nu\lambda}(0, q)$ , i.e., by taking their pure part. In particular,

$$\tilde{H}_\lambda(\mathbf{x}; q) = \sum_{\nu} \tilde{K}_{\nu\lambda}(q) s_\nu(\mathbf{x}). \quad (2.1.5)$$

For a partition  $\lambda$ , we denote by  $\chi^\lambda$  the corresponding irreducible character of  $S_{|\lambda|}$  as in Macdonald [24]. Under this parameterization, the character  $\chi^{(1^n)}$  is the sign character of  $S_{|\lambda|}$  and  $\chi^{(n^1)}$  is the trivial character. Recall also that the decomposition into disjoint cycles provides a natural parameterization of the conjugacy classes of  $S_n$  by the partitions of  $n$ . We then denote by  $\chi_\mu^\lambda$  the value of  $\chi^\lambda$  at the conjugacy class of  $S_{|\lambda|}$  corresponding to  $\mu$  (we use the convention that  $\chi_\mu^\lambda = 0$  if  $|\lambda| \neq |\mu|$ ). The *Green polynomials*  $\{Q_\lambda^\tau(q)\}_{\lambda, \tau \in \mathcal{P}}$  are defined as

$$Q_\lambda^\tau(q) = \sum_{\nu} \chi_\lambda^\nu \tilde{K}_{\nu\tau}(q) \quad (2.1.6)$$

if  $|\lambda| = |\tau|$  and  $Q_\lambda^\tau = 0$  otherwise.

## 2.1.2 Exp and Log

Let  $\Lambda(\mathbf{x}_1, \dots, \mathbf{x}_k) := \Lambda(\mathbf{x}_1) \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} \Lambda(\mathbf{x}_k)$  be the ring of functions separately symmetric in each set  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$  of infinitely many variables. To ease the notation we will simply write  $\Lambda_k$  for the ring  $\Lambda(\mathbf{x}_1, \dots, \mathbf{x}_k) \otimes_{\mathbb{Z}} \mathbb{Q}(q, t)$ .

The power series ring  $\Lambda_k[[T]]$  is endowed with a natural  $\lambda$ -ring structure in which the Adams operations are

$$\psi_d(f(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k, q, t; T)) := f(\mathbf{x}_1^d, \mathbf{x}_2^d, \dots, \mathbf{x}_k^d, q^d, t^d; T^d).$$

Let  $\Lambda_k[[T]]^+$  be the ideal  $T\Lambda_k[[T]]$  of  $\Lambda_k[[T]]$ . Define  $\Psi : \Lambda_k[[T]]^+ \rightarrow \Lambda_k[[T]]^+$  by

$$\Psi(f) := \sum_{n \geq 1} \frac{\psi_n(f)}{n},$$

and  $\text{Exp} : \Lambda_k[[T]]^+ \rightarrow 1 + \Lambda_k[[T]]^+$  by

$$\text{Exp}(f) = \exp(\Psi(f)).$$

The inverse  $\Psi^{-1} : \Lambda_k[[T]]^+ \rightarrow \Lambda_k[[T]]^+$  of  $\Psi$  is given by

$$\Psi^{-1}(f) = \sum_{n \geq 1} \mu(n) \frac{\psi_n(f)}{n}$$

where  $\mu$  is the ordinary Möbius function.

The inverse  $\text{Log} : 1 + \Lambda_k[[T]] \rightarrow \Lambda_k[[T]]$  of  $\text{Exp}$  is given by

$$\text{Log}(f) = \Psi^{-1}(\log(f)).$$

*Remark 2.1.1.* Let  $f = 1 + \sum_{n \geq 1} f_n T^n \in 1 + \Lambda_k[[T]]^+$ . If we write

$$\log(f) = \sum_{n \geq 1} \frac{1}{n} U_n T^n, \quad \text{Log}(f) = \sum_{n \geq 1} V_n T^n,$$

then

$$V_r = \frac{1}{r} \sum_{d|r} \mu(d) \psi_d(U_{r/d}).$$

We have the following propositions (details may be found for instance in Mozgovoy [25]).

For  $g \in \Lambda_k$  and  $n \geq 1$  we put

$$g_n := \frac{1}{n} \sum_{d|n} \mu(d) \psi_{\frac{n}{d}}(g).$$

This is the Möbius inversion formula of  $\psi_n(g) = \sum_{d|n} d \cdot g_d$ .

**Lemma 2.1.2.** *Let  $g \in \Lambda_k$  and  $f_1, f_2 \in 1 + \Lambda_k[[T]]^+$  such that*

$$\log(f_1) = \sum_{d=1}^{\infty} g_d \cdot \log(\psi_d(f_2)).$$

*Then*

$$\text{Log}(f_1) = g \cdot \text{Log}(f_2).$$

**Lemma 2.1.3.** *Assume that  $f \in 1 + \Lambda_k[[T]]^+$  belongs to  $\Lambda(\mathbf{x}_1, \dots, \mathbf{x}_k) \otimes_{\mathbb{Z}} \mathbb{Z}[q, t]$ , then  $\text{Exp}(f) \in \Lambda(\mathbf{x}_1, \dots, \mathbf{x}_k) \otimes_{\mathbb{Z}} \mathbb{Z}[q, t]$ .*

### 2.1.3 Types

We choose once and for all a total ordering  $\geq$  on  $\mathcal{P}$  (e.g. the lexicographic ordering) and we continue to denote by  $\geq$  the total ordering defined on the set of pairs  $\mathbb{Z}_{\geq 0}^* \times \mathcal{P}^*$  as follows: If  $\lambda \neq \mu$  and  $\lambda \geq \mu$ , then  $(d, \lambda) \geq (d', \mu)$ , and  $(d, \lambda) \geq (d', \lambda)$  if  $d \geq d'$ . We denote by  $\mathbf{T}$  the set of non-increasing sequences  $\omega = (d_1, \omega^1) \geq (d_2, \omega^2) \geq \dots \geq (d_r, \omega^r)$ , which we will call a *type*. To alleviate the notation we will then omit the symbol  $\geq$  and write simply  $\omega = (d_1, \omega^1)(d_2, \omega^2) \dots (d_r, \omega^r)$ . The *size* of a type  $\omega$  is  $|\omega| := \sum_i d_i |\lambda^i|$ . We denote by  $\mathbf{T}_n$  the set of types of size  $n$ . We denote by  $m_{d, \lambda}(\omega)$  the multiplicity of  $(d, \lambda)$  in  $\omega$ . As with



partitions it is sometimes convenient to consider a type as a collection of integers  $m_{d,\lambda} \geq 0$  indexed by pairs  $(d, \lambda) \in \mathbb{Z}_{>0} \times \mathcal{P}^*$ . For a type  $\omega = (d_1, \omega^1)(d_2, \omega^2) \cdots (d_r, \omega^r)$ , we put  $n(\omega) = \sum_i d_i n(\omega^i)$  and  $[\omega] := \cup_i d_i \cdot \omega^i$ .

When considering elements  $a_\mu \in \Lambda_k$  indexed by multi-partitions  $\mu = (\mu^1, \dots, \mu^k) \in \mathcal{P}^k$ , we will always assume that they are homogeneous of degree  $(|\mu^1|, \dots, |\mu^k|)$  in the set of variables  $\mathbf{x}_1, \dots, \mathbf{x}_k$ .

Let  $\{a_\mu\}_{\mu \in \mathcal{P}^k}$  be a family of symmetric functions in  $\Lambda_k$  indexed by multi-partitions.

We extend its definition to a *multi-type*  $\omega = (d_1, \omega^1) \cdots (d_s, \omega^s)$  with  $\omega^p \in (\mathcal{P}_{n_p})^k$ , by

$$a_\omega := \prod_p \psi_{d_p}(A_{\omega^p}).$$

For a multi-type  $\omega$  as above, we put

$$C_\omega^o := \begin{cases} \frac{\mu(d)}{d} (-1)^{r-1} \frac{(r-1)!}{\prod_\mu m_{d,\mu}(\omega)!} & \text{if } d_1 = \cdots = d_r = d. \\ 0 & \text{otherwise.} \end{cases}$$

where  $m_{d,\mu}(\omega)$  with  $\mu \in \mathcal{P}^k$  denotes the multiplicity of  $(d, \mu)$  in  $\omega$ .

We have the following lemma (see [10, §2.3.3] for a proof).

**Lemma 2.1.4.** *Let  $\{A_\mu\}_{\mu \in \mathcal{P}^k}$  be a family of symmetric functions in  $\Lambda_k$  with  $A_0 = 1$ . Then*

$$\text{Log} \left( \sum_{\mu \in \mathcal{P}^k} A_\mu T^{|\mu|} \right) = \sum_{\omega} C_\omega^o A_\omega T^{|\omega|} \quad (2.1.7)$$

where  $\omega$  runs over multi-types  $(d_1, \omega^1) \cdots (d_s, \omega^s)$ .

The formal power series  $\sum_{n \geq 0} a_n T^n$  with  $a_n \in \Lambda_k$  that we will consider in what follows will all have  $a_n$  homogeneous of degree  $n$ . Hence we will typically scale the variables of  $\Lambda_k$  by  $1/T$  and eliminate  $T$  altogether.

Given any family  $\{a_\mu\}$  of symmetric functions indexed by partitions  $\mu \in \mathcal{P}$  and a multi-partition  $\mu \in \mathcal{P}^k$  as above define

$$a_\mu := a_{\mu^1}(\mathbf{x}_1) \cdots a_{\mu^k}(\mathbf{x}_k).$$

Let  $\langle \cdot, \cdot \rangle$  be the Hall pairing on  $\Lambda(\mathbf{x})$ , extend its definition to  $\Lambda(\mathbf{x}_1, \dots, \mathbf{x}_k)$  by setting

$$\langle a_1(\mathbf{x}_1) \cdots a_k(\mathbf{x}_k), b_1(\mathbf{x}_1) \cdots b_k(\mathbf{x}_k) \rangle = \langle a_1, b_1 \rangle \cdots \langle a_k, b_k \rangle, \quad (2.1.8)$$

for any  $a_1, \dots, a_k; b_1, \dots, b_k \in \Lambda(\mathbf{x})$  and to formal series by linearity.

### 2.1.4 Cauchy identity

Given a partition  $\lambda \in \mathcal{P}_n$  we define the genus  $g$  hook function  $\mathcal{H}_\lambda(z, w)$  by

$$\mathcal{H}_\lambda(z, w) := \prod_{s \in \lambda} \frac{(z^{2a(s)+1} - w^{2l(s)+1})^{2g}}{(z^{2a(s)+2} - w^{2l(s)})(z^{2a(s)} - w^{2l(s)+2})},$$

where the product is over all cells  $s$  of  $\lambda$  with  $a(s)$  and  $l(s)$  its arm and leg length, respectively. For details on the hook function we refer the reader to [12].

Recall the specialization (cf. [10, §2.3.5])

$$\mathcal{H}_\lambda(0, \sqrt{q}) = \frac{q^{g(\lambda, \lambda)}}{a_\lambda(q)} \quad (2.1.9)$$

where  $a_\lambda(q)$  is the cardinality of the centralizer of a unipotent element of  $\text{GL}_n(\mathbb{F}_q)$  with Jordan form of type  $\lambda$ .

It is also not difficult to verify that the Euler specialization of  $\mathcal{H}_\lambda$  is

$$\mathcal{H}_\lambda(\sqrt{q}, 1/\sqrt{q}) = \left(q^{-\frac{1}{2}\langle\lambda, \lambda\rangle} H_\lambda(q)\right)^{2g-2}. \quad (2.1.10)$$

We have

$$\mathcal{H}_\lambda(z, w) = \mathcal{H}_{\lambda'}(w, z) \text{ and } \mathcal{H}_\lambda(-z, -w) = \mathcal{H}_\lambda(z, w). \quad (2.1.11)$$

Let

$$\Omega(z, w) = \Omega(\mathbf{x}_1, \dots, \mathbf{x}_k; z, w) := \sum_{\lambda \in \mathcal{P}} \mathcal{H}_\lambda(z, w) \prod_{i=1}^k \tilde{H}_\lambda(\mathbf{x}_i; z^2, w^2).$$

By (2.1.1) and (2.1.11) we have

$$\Omega(z, w) = \Omega(w, z) \text{ and } \Omega(-z, -w) = \Omega(z, w). \quad (2.1.12)$$

For  $\mu = (\mu^1, \dots, \mu^k) \in \mathcal{P}^k$ , we let

$$\mathbb{H}_\mu(z, w) := (z^2 - 1)(1 - w^2) \left\langle \text{Log } \Omega(z, w), h_\mu \right\rangle. \quad (2.1.13)$$

By (2.1.12) we have the symmetries

$$\mathbb{H}_\mu(z, w) = \mathbb{H}_\mu(w, z) \text{ and } \mathbb{H}_\mu(-z, -w) = \mathbb{H}_\mu(z, w). \quad (2.1.14)$$

We may recover  $\Omega(z, w)$  from the  $\mathbb{H}_\mu(z, w)$ 's by the formula:

$$\Omega(z, w) = \text{Exp} \left( \sum_{\mu \in \mathcal{P}^k} \frac{\mathbb{H}_\mu(z, w)}{(z^2 - 1)(1 - w^2)} m_\mu \right). \quad (2.1.15)$$

From Formula (2.1.3) and Formula (2.1.10) we have:

**Lemma 2.1.5.** *With the specialization  $y_i = q^{i-1}$ ,*

$$\Omega\left(\sqrt{q}, \frac{1}{\sqrt{q}}\right) = \sum_{\lambda \in \mathcal{P}} q^{(1-q)\langle\lambda, \lambda\rangle} \left(q^{-n(\lambda)} H_\lambda(q)\right)^{2g+k-2} \prod_{i=1}^k s_\lambda(\mathbf{x}_i, \mathbf{y}).$$

**Conjecture 2.1.6.** *The rational function  $\mathbb{H}_\mu(z, w)$  is a polynomial with integer coefficients. It has degree*

$$d_\mu := n^2(2g - 2 + k) - \sum_{i,j} (\mu_j^i)^2 + 2$$

*in each variable and the coefficients of  $\mathbb{H}_\mu(-z, w)$  are non-negative.*

The function  $\mathbb{H}_\mu(z, w)$  is computed in many cases in [10, §1.5].

## 2.2 Characters and Fourier transforms

### 2.2.1 Characters of finite general linear groups

For a finite group  $H$  let us denote by  $\text{Mod}_H$  the category of finite dimensional  $\mathbb{C}[H]$  left modules. Let  $K$  be an other finite group. By an  $H$ -module- $K$  we mean a finite dimensional  $\mathbb{C}$ -vector space  $M$  endowed with a left action of  $H$  and with a right action of  $K$  which commute together. Such a module  $M$  defines a functor  $R_K^H : \text{Mod}_K \rightarrow \text{Mod}_H$  by  $V \mapsto M \otimes_{\mathbb{C}[K]} V$ . Let  $\mathbb{C}(H)$  denotes the  $\mathbb{C}$ -vector space of all functions  $H \rightarrow \mathbb{C}$  which are constant on conjugacy classes. We continue to denote by  $R_K^H$  the  $\mathbb{C}$ -linear map  $\mathbb{C}(K) \rightarrow \mathbb{C}(H)$

induced by the functor  $R_K^H$  (we first define it on irreducible characters and then extend it by linearity to the whole  $\mathbb{C}(K)$ ). Then for any  $f \in \mathbb{C}(K)$ , we have

$$R_K^H(f)(g) = |K|^{-1} \sum_{k \in K} \text{Trace} \left( (g, k^{-1}) | M \right) f(k). \quad (2.2.1)$$

Let  $G = \text{GL}_n(\mathbb{F}_q)$  with  $\mathbb{F}_q$  a finite field. Fix a partition  $\lambda = (\lambda_1, \dots, \lambda_r)$  of  $n$  and let  $\mathcal{F}_\lambda = \mathcal{F}_\lambda(\mathbb{F}_q)$  be the variety of partial flags of  $\mathbb{F}_q$ -vector spaces

$$\{0\} = E^r \subset E^{r-1} \subset \dots \subset E^1 \subset E^0 = (\mathbb{F}_q)^n$$

such that  $\dim(E^{i-1}/E^i) = \lambda_i$ .

Let  $G$  acts on  $\mathcal{F}_\lambda$  in the natural way. Fix an element

$$X_o = (\{0\} = E^r \subset E^{r-1} \subset \dots \subset E^1 \subset E^0 = (\mathbb{F}_q)^n) \in \mathcal{F}_\lambda$$

and denote by  $P_\lambda$  the stabilizer of  $X_o$  in  $G$  and by  $U_\lambda$  the subgroup of elements  $g \in P_\lambda$  which induces the identity on  $E^i/E^{i+1}$  for all  $i = 0, 1, \dots, r-1$ .

Put  $L_\lambda := \text{GL}_{\lambda_r}(\mathbb{F}_q) \times \dots \times \text{GL}_{\lambda_1}(\mathbb{F}_q)$ . Recall that  $U_\lambda$  is a normal subgroup of  $P_\lambda$  and that  $P_\lambda = L_\lambda \ltimes U_\lambda$ .

Denote by  $\mathbb{C}[G/U_\lambda]$  the  $\mathbb{C}$ -vector space generated by the finite set  $G/U_\lambda = \{gU_\lambda \mid g \in G\}$ . The group  $L_\lambda$  (resp.  $G$ ) acts on  $\mathbb{C}[G/U_\lambda]$  as  $(gU_\lambda) \cdot l = glU_\lambda$  (resp. as  $g \cdot (hU_\lambda) = ghU_\lambda$ ). These two actions make  $\mathbb{C}[G/U_\lambda]$  into a  $G$ -module- $L_\lambda$ . The associated functor  $R_{L_\lambda}^G : \text{Mod}_{L_\lambda} \rightarrow \text{Mod}_G$  is the so-called *Harish-Chandra functor*.

We have the following well-known lemma.

**Lemma 2.2.1.** *We denote by 1 the identity character of  $L_\lambda$ . Then for all  $g \in G$ , we have*

$$R_{L_\lambda}^G(1)(g) = \#\{X \in \mathcal{F}_\lambda \mid g \cdot X = X\}.$$

*Proof.* By Formula (2.2.1) we have

$$\begin{aligned} R_{L_\lambda}^G(1)(g) &= |L_\lambda|^{-1} \sum_{k \in L_\lambda} \#\{hU_\lambda \mid ghU_\lambda = hkU_\lambda\} \\ &= |L_\lambda|^{-1} \sum_{k \in L_\lambda} \#\{hU_\lambda \mid gh \in hkU_\lambda\} \\ &= |L_\lambda|^{-1} \#\{hU_\lambda \mid gh \in hP_\lambda\} \\ &= \#\{hP_\lambda \mid ghP_\lambda = hP_\lambda\}. \end{aligned}$$

We deduce the lemma from last equality by noticing that the map  $G \rightarrow \mathcal{F}_\lambda$ ,  $g \mapsto g \cdot X_o$  induces a bijection  $G/P_\lambda \rightarrow \mathcal{F}_\lambda$ .  $\square$

We now recall the definition of the type of a conjugacy class  $C$  of  $G$  (cf. [10, 4.1]). The Frobenius  $f : \mathbb{F} \rightarrow \mathbb{F}$ ,  $x \mapsto x^q$  acts on the set of eigenvalues of  $C$ . Let us write the set of eigenvalues of  $C$  as a disjoint union

$$\{\gamma_1, \gamma_1^q, \dots\} \bigsqcup \{\gamma_2, \gamma_2^q, \dots\} \bigsqcup \dots \bigsqcup \{\gamma_r, \gamma_r^q, \dots\}$$

of  $\langle f \rangle$ -orbits, and let  $m_i$  be the multiplicity of  $\gamma_i$ . The unipotent part of an element of  $C$  defines a unique partition  $\omega^i$  of  $m_i$ . Re-ordering if necessary we may assume that  $(d_1, \omega^1) \geq (d_2, \omega^2) \geq \dots \geq (d_r, \omega^r)$ . We then call  $\omega = (d_1, \omega^1) \dots (d_r, \omega^r) \in \mathbf{T}_n$  the *type* of  $C$ .

Put  $T := L_{(1,1,\dots,1)}$ . It is the subgroup of diagonal matrices of  $G$ . The decomposition of  $R_T^{L_\lambda}(1)$  as a sum of irreducible characters reads

$$R_T^{L_\lambda}(1) = \sum_{\chi \in \text{Irr}(W_{L_\lambda})} \chi(1) \cdot \mathcal{U}_\chi,$$

where  $W_{L_\lambda} := N_{L_\lambda}(T)/T$  is the Weyl group of  $L_\lambda$ . We call the irreducible characters  $\{\mathcal{U}_\chi\}_\chi$  the *unipotent* characters of  $L_\lambda$ . The character  $\mathcal{U}_1$  is the trivial character of  $L_\lambda$ . Since  $W_{L_\lambda} \simeq S_{\lambda_1} \times \dots \times S_{\lambda_r}$ , the irreducible characters of  $W_{L_\lambda}$  are  $\chi^\tau := \chi^{\tau^1} \cdots \chi^{\tau^r}$  where  $\tau$  runs over the set of types  $\tau = \{(1, \tau^i)\}_{i=1,\dots,r}$  with  $\tau^i$  a partition of  $\lambda_i$ . We denote by  $\mathcal{U}_\tau$  the unipotent character of  $L_\lambda$  corresponding to such a type  $\tau$ .

**Theorem 2.2.2.** *Let  $\mathcal{U}_\tau$  be a unipotent character of  $L_\lambda$  and let  $C$  be a conjugacy class of type  $\omega$ . Then*

$$R_{L_\lambda}^G(\mathcal{U}_\tau)(C) = \langle \tilde{H}_\omega(\mathbf{x}, q), s_\tau(\mathbf{x}) \rangle.$$

*Proof.* The proof is contained in [10] although the formula is not explicitly written there. For the convenience of the reader we now explain how to extract the proof from [10]. For  $w \in W_\lambda$ , we denote by  $R_{T_w}^G(1)$  the corresponding *Deligne-Lusztig character* of  $G$ . Its construction is outlined in [10, 2.6.4]. The character  $\mathcal{U}_\tau$  of  $L_\lambda$  decomposes as,

$$\mathcal{U}_\tau = |W_\lambda|^{-1} \sum_{w \in W_\lambda} \chi_w^\tau \cdot R_{T_w}^{L_\lambda}(1)$$

where  $\chi_w^\tau$  denotes the value of  $\chi^\tau$  at  $w$ . Applying the Harish-Chandra induction  $R_{L_\lambda}^G$  to both side and using the transitivity of induction we find that

$$R_{L_\lambda}^G(\mathcal{U}_\tau) = |W_\lambda|^{-1} \sum_{w \in W_\lambda} \chi_w^\tau \cdot R_{T_w}^G(1).$$

We are now in position to use the calculation in [10]. Notice that the right handside of the above formula is the right hand side of the first formula displayed in the proof of [10, Theorem 4.3.1] with  $(M, \theta^{T_w}, \tilde{\varphi}) = (L_\lambda, 1, \chi^\tau)$  and so the same calculation to get [10, (4.3.2)] together with [10, (4.3.3)] gives in our case

$$R_{L_\lambda}^G(\mathcal{U}_\tau)(C) = \sum_{\alpha} z_\alpha^{-1} \chi_\alpha^\tau \sum_{\{\beta \mid [\beta] = [\alpha]\}} \mathcal{Q}_\beta^\omega(q) z_{[\alpha]} z_\beta^{-1}$$

where the notation are those of [10, 4.3]. We now apply [10, Lemma 2.3.5] to get

$$R_{L_\lambda}^G(\mathcal{U}_\tau)(C) = \langle \tilde{H}_\omega(\mathbf{x}; q), s_\tau(\mathbf{x}) \rangle.$$

□

If  $\alpha$  is the type  $(1, (\lambda_1)) \cdots (1, (\lambda_r))$ , then  $s_\alpha(\mathbf{x}) = h_\lambda(\mathbf{x})$ . Hence we have:

**Corollary 2.2.3.** *If  $C$  is a conjugacy class of  $G$  type  $\omega$ , then*

$$R_{L_\lambda}^G(1)(C) = \langle \tilde{H}_\omega(\mathbf{x}, q), h_\lambda(\mathbf{x}) \rangle.$$

**Corollary 2.2.4.** *Put  $\mathcal{F}_{\lambda, \omega}^\#(q) := \#\{X \in \mathcal{F}_\lambda \mid g \cdot X = X\}$  where  $g \in G$  is an element in a conjugacy class of type  $\omega$ . Then*

$$\tilde{H}_\omega(\mathbf{x}, q) = \sum_{\lambda} \mathcal{F}_{\lambda, \omega}^\#(q) m_\lambda(\mathbf{x}).$$

*Proof.* It follows from Lemma 2.2.1 and Corollary 2.2.3. □

We now recall how to construct from a partition  $\lambda = (\lambda_1, \dots, \lambda_r)$  of  $n$  a certain family of irreducible characters of  $G$ . Choose  $r$  distinct linear character  $\alpha_1, \dots, \alpha_r$  of  $\mathbb{F}_q^\times$ . This defines for each  $i$  a linear character  $\tilde{\alpha}_i : \mathrm{GL}_{\lambda_i}(\mathbb{F}_q) \rightarrow \mathbb{C}^\times$ ,  $g \mapsto \alpha_i(\det(g))$ , and hence a linear character  $\tilde{\alpha} : L_\lambda \rightarrow \mathbb{C}^\times$ ,  $(g_i) \mapsto \tilde{\alpha}_r(g_r) \cdots \tilde{\alpha}_1(g_1)$ . This linear character has the following property: for an element  $g \in N_G(L_\lambda)$ , we have  $\tilde{\alpha}(g^{-1}lg) = \tilde{\alpha}(l)$  for all  $l \in L_\lambda$  if and only if  $g \in L_\lambda$ . A linear character of  $L_\lambda$  which satisfies this property is called a *regular* character of  $L_\lambda$ .

It is a well-known fact that  $R_{L_\lambda}^G(\tilde{\alpha})$  is an irreducible character of  $G$ . Note that the irreducible characters of  $G$  are not all obtained in this way (see [23] for the complete description of the irreducible characters of  $G$  in terms of Deligne-Luzstig induction).

We now recall the definition of generic tuples of irreducible characters (cf. [10, Definition 4.2.2]). Since in this paper we are only considering irreducible characters of the form  $R_{L_\lambda}^G(\tilde{\alpha})$ , the definition given in [10, Definition 4.2.2] simplifies.

**Definition 2.2.5.** Consider irreducible characters  $R_{L_{\lambda^1}}^G(\tilde{\alpha}_1), \dots, R_{L_{\lambda^k}}^G(\tilde{\alpha}_k)$  of  $G$  as above for a multi-partition  $\lambda = (\lambda^1, \dots, \lambda^k) \in (\mathcal{P}_n)^k$ . Let  $T$  be the subgroup of  $G$  of diagonal matrices. Note that  $T \subset L_\lambda$  for all partition  $\lambda$ , and so  $T$  contains the center  $Z_\lambda$  of any  $L_\lambda$ . Consider the linear character  $\alpha = (\tilde{\alpha}_1|_T) \cdots (\tilde{\alpha}_k|_T)$  of  $T$ . Then we say that the tuple  $(R_{L_{\lambda^1}}^G(\tilde{\alpha}_1), \dots, R_{L_{\lambda^k}}^G(\tilde{\alpha}_k))$  is *generic* if the restriction  $\alpha|_{Z_\lambda}$  of  $\alpha$  to any subtorus  $Z_\lambda$ , with  $\lambda \in \mathcal{P}_n - \{(n)\}$ , is non-trivial and if  $\alpha|_{Z_{(n)}}$  is trivial (the center  $Z_{(n)} \simeq \mathbb{F}_q^\times$  consists of scalar matrices  $a \cdot I_n$ ).

We can show as for conjugacy classes [10, Lemma 2.1.2] that if the characteristic  $p$  of  $\mathbb{F}_q$  and  $q$  are sufficiently large, generic tuples of irreducible characters of a given type  $\lambda$  always exist.

Put  $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{F}_q)$ . For  $X \in \mathfrak{g}$ , put

$$\Lambda^1(X) := \#\{Y \in \mathfrak{g} \mid [X, Y] = 0\}.$$

The restriction  $\Lambda^1 : G \rightarrow \mathbb{C}$  of  $\Lambda^1$  to  $G \subset \mathfrak{g}$  is the character of the representation  $G \rightarrow \mathrm{GL}(\mathbb{C}[\mathfrak{g}])$  induced by the conjugation action of  $G$  on  $\mathfrak{g}$ . Fix a non-negative integer  $g$  and put  $\Lambda := (\Lambda^1)^{\otimes g}$ .

For a multi-partition  $\mu = (\mu^1, \dots, \mu^k) \in (\mathcal{P}_n)^k$  and a generic tuple  $(R_{L_{\mu^1}}^G(\tilde{\alpha}_1), \dots, R_{L_{\mu^k}}^G(\tilde{\alpha}_k))$  of irreducible characters we put

$$R_\mu := R_{L_{\mu^1}}^G(\tilde{\alpha}_1) \otimes \cdots \otimes R_{L_{\mu^k}}^G(\tilde{\alpha}_k).$$

For two class functions  $f, g \in \mathbb{C}(G)$ , we define

$$\langle f, g \rangle := |G|^{-1} \sum_{h \in G} f(h) \overline{g(h)}.$$

We have the following theorem [10, Theorem 1.4.1].

**Theorem 2.2.6.** *We have*

$$\langle \Lambda \otimes R_\mu, 1 \rangle = \mathbb{H}_\mu(0, \sqrt{q})$$

where  $\mathbb{H}_\mu(z, w)$  is the function defined in §2.1.4.

**Corollary 2.2.7.** *The multiplicity  $\langle \Lambda \otimes R_\mu, 1 \rangle$  depends only on  $\mu$  and not on the choice of linear characters  $(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k)$ .*

## 2.2.2 Fourier transforms

Let  $\mathrm{Fun}(\mathfrak{g})$  be the  $\mathbb{C}$ -vector space of all functions  $\mathfrak{g} \rightarrow \mathbb{C}$  and by  $\mathbb{C}(\mathfrak{g})$  the subspace of functions  $\mathfrak{g} \rightarrow \mathbb{C}$  which are constant on  $G$ -orbits of  $\mathfrak{g}$  for the conjugation action of  $G$  on  $\mathfrak{g}$ .

Let  $\Psi : \mathbb{F}_q \rightarrow \mathbb{C}^\times$  be a non-trivial additive character and consider the trace pairing  $\mathrm{Tr} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}^\times$ . Define the Fourier transform  $\mathcal{F}^\Psi : \mathrm{Fun}(\mathfrak{g}) \rightarrow \mathrm{Fun}(\mathfrak{g})$  by the formula

$$\mathcal{F}^{\mathfrak{g}}(f)(x) = \sum_{y \in \mathfrak{g}} \Psi(\text{Tr}(xy)) f(y)$$

for all  $f \in \text{Fun}(\mathfrak{g})$  and  $x \in \mathfrak{g}$ .

The Fourier transform satisfies the following easy property.

**Proposition 2.2.8.** *For any  $f \in \text{Fun}(\mathfrak{g})$  we have:*

$$|\mathfrak{g}| \cdot f(0) = \sum_{x \in \mathfrak{g}} \mathcal{F}^{\mathfrak{g}}(f)(x).$$

Let  $*$  be the convolution product on  $\text{Fun}(\mathfrak{g})$  defined by

$$(f * g)(a) = \sum_{x+y=a} f(x)g(y)$$

for any two functions  $f, g \in \text{Fun}(\mathfrak{g})$ .

Recall that

$$\mathcal{F}^{\mathfrak{g}}(f * g) = \mathcal{F}^{\mathfrak{g}}(f) \cdot \mathcal{F}^{\mathfrak{g}}(g). \quad (2.2.2)$$

For a partition  $\lambda$  of  $n$ , let  $\mathfrak{p}_\lambda$ ,  $\mathfrak{l}_\lambda$ ,  $\mathfrak{u}_\lambda$  be the Lie sub-algebras of  $\mathfrak{g}$  corresponding respectively to the subgroups  $P_\lambda$ ,  $L_\lambda$ ,  $U_\lambda$  defined in §2.2, namely  $\mathfrak{l}_\lambda = \bigoplus_i \mathfrak{gl}_{\lambda_i}(\mathbb{F}_q)$ ,  $\mathfrak{p}_\lambda$  is the parabolic sub-algebra of  $\mathfrak{g}$  having  $\mathfrak{l}_\lambda$  as a Levi sub-algebra and containing the upper triangular matrices. We have  $\mathfrak{p}_\lambda = \mathfrak{l}_\lambda \oplus \mathfrak{u}_\lambda$ .

Define the two functions  $R_{\mathfrak{l}_\lambda}^{\mathfrak{g}}(1), Q_{\mathfrak{l}_\lambda}^{\mathfrak{g}} \in \mathbb{C}(\mathfrak{g})$  by

$$\begin{aligned} R_{\mathfrak{l}_\lambda}^{\mathfrak{g}}(1)(x) &= |P_\lambda|^{-1} \#\{g \in G \mid g^{-1}xg \in \mathfrak{p}_\lambda\}, \\ Q_{\mathfrak{l}_\lambda}^{\mathfrak{g}}(x) &= |P_\lambda|^{-1} \#\{g \in G \mid g^{-1}xg \in \mathfrak{u}_\lambda\}. \end{aligned}$$

We define the type of a  $G$ -orbit of  $\mathfrak{g}$  similarly as in the group setting (see above Corollary 2.2.3). The types of the  $G$ -orbits of  $\mathfrak{g}$  are then also parameterized by  $\mathbf{T}_n$ .

*Remark 2.2.9.* From Lemma 2.2.1, we see that  $R_{L_\lambda}^G(1)(x) = |P_\lambda|^{-1} \#\{g \in G \mid g^{-1}xg \in P_\lambda\}$ , hence  $R_{\mathfrak{l}_\lambda}^{\mathfrak{g}}(1)$  is the Lie algebra analogue of  $R_{L_\lambda}^G(1)$  and the two functions take the same values on elements of same type.

**Proposition 2.2.10.** *We have*

$$\mathcal{F}^{\mathfrak{g}}(Q_{\mathfrak{l}_\lambda}^{\mathfrak{g}}) = q^{\frac{1}{2}(n^2 - \sum_i \lambda_i^2)} R_{\mathfrak{l}_\lambda}^{\mathfrak{g}}(1).$$

*Proof.* Consider the  $\mathbb{C}$ -linear map  $R_{\mathfrak{l}_\lambda}^{\mathfrak{g}} : \mathbb{C}(\mathfrak{l}_\lambda) \rightarrow \mathbb{C}(\mathfrak{g})$  defined by

$$R_{\mathfrak{l}_\lambda}^{\mathfrak{g}}(f)(x) = |P_\lambda|^{-1} \sum_{\{g \in G \mid g^{-1}xg \in \mathfrak{p}_\lambda\}} f(\pi(g^{-1}xg))$$

where  $\pi : \mathfrak{p}_\lambda \rightarrow \mathfrak{l}_\lambda$  is the canonical projection. Then it is easy to see that  $Q_{\mathfrak{l}_\lambda}^{\mathfrak{g}} = R_{\mathfrak{l}_\lambda}^{\mathfrak{g}}(1_0)$  where  $1_0 \in \mathbb{C}(\mathfrak{l}_\lambda)$  is the characteristic function of  $0 \in \mathfrak{l}_\lambda$ , i.e.,  $1_0(x) = 1$  if  $x = 0$  and  $1_0(x) = 0$  otherwise. The result follows from the easy fact that  $\mathcal{F}^{\mathfrak{l}_\lambda}(1_0)$  is the identity function 1 on  $\mathfrak{l}_\lambda$  and the fact (see Lehrer [21]) that

$$\mathcal{F}^{\mathfrak{g}} \circ R_{\mathfrak{l}_\lambda}^{\mathfrak{g}} = q^{\frac{1}{2}(n^2 - \sum_i \lambda_i^2)} R_{\mathfrak{l}_\lambda}^{\mathfrak{g}} \circ \mathcal{F}^{\mathfrak{l}_\lambda}.$$

□

*Remark 2.2.11.* For  $x \in \mathfrak{g}$ , denote by  $1_x \in \text{Fun}(\mathfrak{g})$  the characteristic function of  $x$  that takes the value 1 at  $x$  and the value 0 elsewhere. Note that  $\mathcal{F}^{\mathfrak{g}}(1_x)$  is the linear character  $\mathfrak{g} \rightarrow \mathbb{C}$ ,  $t \mapsto \Psi(\text{Tr}(xt))$  of the abelian group  $(\mathfrak{g}, +)$ . Hence if  $f : \mathfrak{g} \rightarrow \mathbb{C}$  is a function which takes integer values, then  $\mathcal{F}^{\mathfrak{g}}(f)$  is a character (not necessarily irreducible) of  $(\mathfrak{g}, +)$ . Since the Green functions  $Q_{\mathfrak{l}_\lambda}^{\mathfrak{g}}$  take integer values, by Proposition 2.2.10 the function  $q^{\frac{1}{2}(n^2 - \sum_i \lambda_i^2)} R_{\mathfrak{l}_\lambda}^{\mathfrak{g}}(1)$  is a character of  $(\mathfrak{g}, +)$ .

### 3 Absolutely indecomposable representations

#### 3.1 Generalities on quiver representations

Let  $\Gamma$  be a finite quiver,  $I$  be the set of its vertices and let  $\Omega$  be the set of its arrows. For  $\gamma \in \Omega$ , we denote by  $h(\gamma), t(\gamma) \in I$  the head and the tail of  $\gamma$ . A *dimension vector* of  $\Gamma$  is a collection of non-negative integers  $\mathbf{v} = \{v_i\}_{i \in I}$  and a *representation*  $\varphi$  of  $\Gamma$  of dimension  $\mathbf{v}$  over a field  $\mathbb{K}$  is a collection of  $\mathbb{K}$ -linear maps  $\varphi = \{\varphi_\gamma : V_{t(\gamma)} \rightarrow V_{h(\gamma)}\}_{\gamma \in \Omega}$  with  $\dim V_i = v_i$ . Let  $\text{Rep}_{\Gamma, \mathbf{v}}(\mathbb{K})$  be the  $\mathbb{K}$ -vector space of all representations of  $\Gamma$  of dimension  $\mathbf{v}$  over  $\mathbb{K}$ . If  $\varphi \in \text{Rep}_{\Gamma, \mathbf{v}}(\mathbb{K})$ ,  $\varphi' \in \text{Rep}_{\Gamma, \mathbf{v}'}(\mathbb{K})$ , then a morphism  $f : \varphi \rightarrow \varphi'$  is a collection of  $\mathbb{K}$ -linear maps  $f_i : V_i \rightarrow V'_i$ ,  $i \in I$  such that for all  $\gamma \in \Omega$ , we have  $f_{h(\gamma)} \circ \varphi_\gamma = \varphi'_\gamma \circ f_{t(\gamma)}$ .

We define in the obvious way direct sums  $\varphi \oplus \varphi' \in \text{Rep}_{\mathbb{K}}(\Gamma, \mathbf{v} + \mathbf{v}')$  of representations. A representation of  $\Gamma$  is said to be *indecomposable* over  $\mathbb{K}$  if it is not isomorphic to a direct sum of two non-zero representations of  $\Gamma$ . If an indecomposable representation of  $\Gamma$  remains indecomposable over any finite extension of  $\mathbb{K}$ , we say that it is *absolutely indecomposable*. Denote by  $M_{\Gamma, \mathbf{v}}(\mathbb{K})$  be the set of isomorphism classes of  $\text{Rep}_{\Gamma, \mathbf{v}}(\mathbb{K})$  and by  $A_{\Gamma, \mathbf{v}}(\mathbb{K})$  the subset of absolutely indecomposable representations of  $\text{Rep}_{\Gamma, \mathbf{v}}(\mathbb{K})$ .

By a theorem of Kac there exists a polynomial  $A_{\Gamma, \mathbf{v}}(T) \in \mathbb{Z}[T]$  such that for any finite field with  $q$  elements  $A_{\Gamma, \mathbf{v}}(q) = \#A_{\Gamma, \mathbf{v}}(\mathbb{F}_q)$ . We call  $A_{\Gamma, \mathbf{v}}$  the *A-polynomial* of  $(\Gamma, \mathbf{v})$ .

Let  $\Phi(\Gamma) \subset \mathbb{Z}^I$  be the root system associated with the quiver  $\Gamma$  following Kac [16] and let  $\Phi(\Gamma)^+ \subset (\mathbb{Z}_{\geq 0})^I$  be the subset of positive roots. Let  $\mathbf{C} = (c_{ij})_{i, j}$  be the Cartan matrix of  $\Gamma$ , namely

$$c_{ij} = \begin{cases} 2 - 2(\text{the number of edges joining } i \text{ to itself}) & \text{if } i = j \\ -(\text{the number of edges joining } i \text{ to } j) & \text{otherwise.} \end{cases}$$

Then we have the following well-known theorem (see Kac [16]).

**Theorem 3.1.1.**  $A_{\Gamma, \mathbf{v}}(q) \neq 0$  if and only if  $\mathbf{v} \in \Phi(\Gamma)^+$ ;  $A_{\Gamma, \mathbf{v}}(q) = 1$  if and only if  $\mathbf{v}$  is a real root. The polynomial  $A_{\Gamma, \mathbf{v}}$ , if non-zero, is monic of degree  $2 - {}^t \mathbf{v} \mathbf{C} \mathbf{v}$ .

We have the following conjecture due to Kac [16].

**Conjecture 3.1.2.** The polynomial  $A_{\Gamma, \mathbf{v}}(T)$  has non-negative coefficients.

We will say that a dimension vector  $\mathbf{v}$  is *indivisible* if  $\gcd\{v_i\}_{i \in I} = 1$ . Conjecture 3.1.2 was proved by Crawley-Boevey and van den Bergh [1] when the dimension vector  $\mathbf{v}$  is indivisible. This was achieved by giving a cohomological interpretation of  $A_{\Gamma, \mathbf{v}}(q)$ . A more recent work by Mozgovoy [26] proves Conjecture 3.1.2 for any dimension vector for quivers with at least one loop at each vertex. His proof is accomplished via work of Kontsevich-Soibelman [18] and Efimov [4] on motivic Donaldson-Thomas invariants associated to quivers.

By Kac[16], there exists a polynomial  $M_{\Gamma, \mathbf{v}}(q) \in \mathbb{Q}[T]$  such that  $M_{\Gamma, \mathbf{v}}(q) := \#M_{\Gamma, \mathbf{v}}(\mathbb{F}_q)$  for any finite field  $\mathbb{F}_q$ . The following formula is a reformation of Hua's formula [14].

**Theorem 3.1.3.** We have

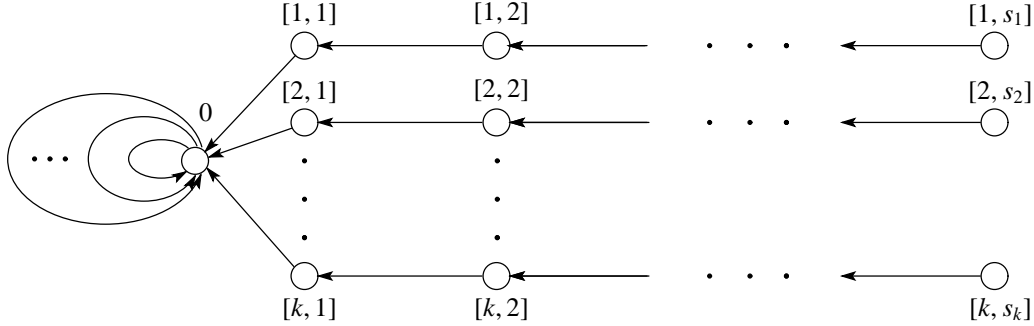
$$\text{Log} \left( \sum_{\mathbf{v} \in (\mathbb{Z}_{\geq 0})^I} M_{\Gamma, \mathbf{v}}(q) X^{\mathbf{v}} \right) = \sum_{\mathbf{v} \in (\mathbb{Z}_{\geq 0})^I - \{0\}} A_{\Gamma, \mathbf{v}}(q) X^{\mathbf{v}},$$

where  $X^{\mathbf{v}}$  is the monomial  $\prod_{i \in I} X_i^{v_i}$  for some independent commuting variables  $\{X_i\}_{i \in I}$ .

Since  $A_{\Gamma, \mathbf{v}}(q) \in \mathbb{Z}[q]$ , we see by Theorem 3.1.3 and Lemma 2.1.3, that  $M_{\Gamma, \mathbf{v}}(q)$  also has integer coefficients.

### 3.2 Comet-shaped quivers

Fix strictly positive integers  $g, k, s_1, \dots, s_k$  and consider the following (comet-shaped) quiver  $\Gamma$  with  $g$  loops on the central vertex and with set of vertices  $I = \{0\} \cup \{[i, j] \mid i = 1, \dots, k; j = 1, \dots, s_i\}$ .



Let  $\Omega^0$  denote the set of arrows  $\gamma \in \Omega$  such that  $h(\gamma) \neq t(\gamma)$ .

**Lemma 3.2.1.** *Let  $\mathbb{K}$  be any field. Let  $\varphi \in \text{Rep}_{\Gamma, \mathbf{v}}(\mathbb{K})$  and assume that  $v_0 > 0$ . If  $\varphi$  is indecomposable, then the linear maps  $\varphi_\gamma$ , with  $\gamma \in \Omega^0$ , are all injective.*

*Proof.* If  $\gamma$  is the arrow  $[i, j] \rightarrow [i, j-1]$ , with  $j = 1, \dots, s_i$  and with the convention that  $[i, 0] = 0$ , we use the notation  $\varphi_{ij} : V_{[i, j]} \rightarrow V_{[i, j-1]}$  rather than  $\varphi_\gamma : V_{t(\gamma)} \rightarrow V_{h(\gamma)}$ . Assume that  $\varphi_{ij}$  is not injective. We define a graded vector subspace  $V' = \bigoplus_{i \in I} V'_i$  of  $V = \bigoplus_{i \in I} V_i$  as follows.

If the vertex  $i$  is not one of the vertices  $[i, j], [i, j+1], \dots, [i, s_i]$ , we put  $V'_i := \{0\}$ . We put  $V'_{[i, j]} := \text{Ker } \varphi_{ij}$ ,  $V'_{[i, j+1]} := \varphi_{i(j+1)}^{-1}(V'_{[i, j]})$ ,  $\dots$ ,  $V'_{[i, s_i]} := \varphi_{i s_i}^{-1}(V'_{[i, s_i-1]})$ . Let  $\mathbf{v}'$  be the dimension of the graded space  $V' = \bigoplus_{i \in I} V'_i$  which we consider as a dimension vector of  $\Gamma$ . Define  $\varphi' \in \text{Rep}_{\Gamma, \mathbf{v}'}(\mathbb{K})$  as the restriction of  $\varphi$  to  $V'$ . It is a non-zero subrepresentation of  $\varphi$ . It is now possible to define a graded vector subspace  $V'' = \bigoplus_{i \in I} V''_i$  of  $V$  such that the restriction  $\varphi''$  of  $\varphi$  to  $V''$  satisfies  $\varphi = \varphi'' \oplus \varphi'$ : we start by taking any subspace  $V''_{[i, j]}$  such that  $V_{[i, j]} = V'_{[i, j]} \oplus V''_{[i, j]}$ , then define  $V''_{[i, j+r]}$  from  $V''_{[i, j]}$  as  $V'_{[i, j+r]}$  was defined from  $V_{[i, j]}$ , and finally put  $V''_i := V_i$  if the vertex  $i$  is not one of the vertices  $[i, j], [i, j+1], \dots, [i, s_i]$ . As  $v_0 > 0$ , the subrepresentation  $\varphi''$  is non-zero, and so  $\varphi$  is not indecomposable.  $\square$

We denote by  $\text{Rep}_{\Gamma, \mathbf{v}}^*(\mathbb{F}_q)$  be the subspace of representation  $\varphi \in \text{Rep}_{\Gamma, \mathbf{v}}(\mathbb{F}_q)$  such that  $\varphi_\gamma$  is injective for all  $\gamma \in \Omega^0$ , and by  $M_{\Gamma, \mathbf{v}}^*(\mathbb{F}_q)$  the set of isomorphism classes of  $\text{Rep}_{\Gamma, \mathbf{v}}^*(\mathbb{F}_q)$ . Put  $M_{\Gamma, \mathbf{v}}^*(q) = \#\{M_{\Gamma, \mathbf{v}}^*(\mathbb{F}_q)\}$ . Following [2] we say that a dimension vector  $\mathbf{v}$  of  $\Gamma$  is *strict* if for each  $i = 1, \dots, k$  we have  $n_0 \geq v_{[i, 1]} \geq v_{[i, 2]} \geq \dots \geq v_{[i, s_i]}$ . Let us denote by  $\mathcal{S}$  the set of strict dimension vector of  $\Gamma$ .

**Proposition 3.2.2.**

$$\text{Log} \left( \sum_{\mathbf{v} \in \mathcal{S}} M_{\Gamma, \mathbf{v}}^*(q) X^{\mathbf{v}} \right) = \sum_{\mathbf{v} \in \mathcal{S} - \{0\}} A_{\Gamma, \mathbf{v}}(q) X^{\mathbf{v}}.$$

*Proof.* Let us denote by  $I_{\Gamma, \mathbf{v}}(q)$  the number of isomorphism classes of indecomposable representations in  $\text{Rep}_{\Gamma, \mathbf{v}}(\mathbb{F}_q)$ . By the Krull-Schmidt theorem, a representation of  $\Gamma$  decomposes as a direct sum of indecomposable representation in a unique way up to permutation of the summands. Notice that, for  $\mathbf{v} \in \mathcal{S}$ , each summand of an element of  $\text{Rep}_{\Gamma, \mathbf{v}}^*(\mathbb{F}_q)$  lives in some  $\text{Rep}_{\Gamma, \mathbf{w}}^*(\mathbb{F}_q)$  for some  $\mathbf{w} \in \mathcal{S}$ . On the other hand, by Lemma 3.2.1,  $\text{Rep}_{\Gamma, \mathbf{v}}^*(\mathbb{F}_q)$  contains all the indecomposable representations in  $\text{Rep}_{\Gamma, \mathbf{v}}(\mathbb{F}_q)$ . This implies the following identity

$$\sum_{\mathbf{v} \in \mathcal{S}} M_{\Gamma, \mathbf{v}}^*(q) X^{\mathbf{v}} = \prod_{\mathbf{v} \in \mathcal{S} - \{0\}} (1 - X^{\mathbf{v}})^{-I_{\Gamma, \mathbf{v}}(q)},$$



where  $X^\mathbf{v}$  denotes the monomial  $\prod_{i \in I} X_i^{v_i}$  for some fixed independent commuting variables  $\{X_i\}_{i \in I}$ . Exactly as Hua [14, Proof of Lemma 4.5] does we show from this formal identity that

$$\text{Log} \left( \sum_{\mathbf{v} \in S} M_{\Gamma, \mathbf{v}}^*(q) X^\mathbf{v} \right) = \sum_{\mathbf{v} \in S - \{0\}} A_{\Gamma, \mathbf{v}}(q) X^\mathbf{v}.$$

□

It follows from Proposition 3.2.2 that since  $A_{\Gamma, \mathbf{v}}(T) \in \mathbb{Z}[T]$  the quantity  $M_{\Gamma, \mathbf{v}}^*(q)$  is also the evaluation of a polynomial with integer coefficients at  $T = q$ .

Given a non-increasing sequence  $u = (n_0 \geq n_1 \geq \dots)$  of non-negative integers we let  $\Delta u$  be the sequence of successive differences  $n_0 - n_1, n_1 - n_2, \dots$ . We extend the notation of §2.2.1 and denote by  $\mathcal{F}_{\Delta u}$  the set of partial flags of  $\mathbb{F}_q$ -vector spaces

$$\{0\} \subseteq E^r \subseteq \dots \subseteq E^1 \subseteq E^0 = (\mathbb{F}_q)^{n_0}$$

such that  $\dim(E^i) = n_i$ .

Assume that  $\mathbf{v} \in S$  and let  $\mu = (\mu^1, \dots, \mu^k)$ , where  $\mu^i$  is the partition obtained from  $\Delta \mathbf{v}_i$  by reordering, where  $\mathbf{v}_i := (v_0 \geq v_{[i,1]} \geq \dots \geq v_{[i,s_i]})$ . Consider the set of orbits

$$\mathfrak{G}_\mu(\mathbb{F}_q) := \left( \text{Mat}_{n_0}(\mathbb{F}_q)^g \times \prod_{i=1}^k \mathcal{F}_{\mu^i}(\mathbb{F}_q) \right) \Big/ \text{GL}_{v_0}(\mathbb{F}_q),$$

where  $\text{GL}_{v_0}(\mathbb{F}_q)$  acts by conjugation on the first  $g$  coordinates and in the obvious way on each  $\mathcal{F}_{\mu^i}(\mathbb{F}_q)$ .

Let  $\varphi \in \text{Rep}_{\Gamma, \mathbf{v}}^*(\mathbb{F}_q)$  with underlying graded vector space  $V = V_0 \oplus \bigoplus_{i,j} V_{[i,j]}$ . We choose a basis of  $V_0$  and we identify  $V_0$  with  $(\mathbb{F}_q)^{v_0}$ . In the chosen basis, the  $g$  maps  $\varphi_\gamma$ , with  $\gamma \in \Omega - \Omega^0$ , give an element in  $\text{Mat}_{v_0}(\mathbb{F}_q)^g$ . For each  $i = 1, \dots, k$ , we obtain a partial flag by taking the images in  $(\mathbb{F}_q)^{v_0}$  of the  $V_{[i,j]}$ 's via the compositions of the  $\varphi_\gamma$ 's where  $\gamma$  runs over the arrows of the  $i$ -th leg of  $\Gamma$ . We thus have defined a map

$$\text{Rep}_{\Gamma, \mathbf{v}}^*(\mathbb{F}_q) \longrightarrow \left( \text{Mat}_{v_0}(\mathbb{F}_q)^g \times \prod_{i=1}^k \mathcal{F}_{\Delta \mathbf{v}_i}(\mathbb{F}_q) \right) \Big/ \text{GL}_{v_0}(\mathbb{F}_q).$$

The target set is clearly in bijection with  $\mathfrak{G}_\mu(\mathbb{F}_q)$  and two elements of  $\text{Rep}_{\Gamma, \mathbf{v}}^*(\mathbb{F}_q)$  have the same image if and only if they are isomorphic. Hence this map induces an isomorphism  $\text{M}_{\Gamma, \mathbf{v}}^*(\mathbb{F}_q) \simeq \mathfrak{G}_\mu(\mathbb{F}_q)$ .

We define a new comet-shaped quiver  $\Gamma_\mu$  consisting of  $g$  loops on a central vertex and  $k$  legs of length  $l(\mu^i) - 1$  and let  $\mathbf{v}_\mu$  be the dimension vector as in §1.3. Applying the above construction to the pair  $\Gamma_\mu, \mathbf{v}_\mu$  we obtain a bijection  $\text{M}_{\Gamma_\mu, \mathbf{v}_\mu}^*(\mathbb{F}_q) \simeq \mathfrak{G}_\mu(\mathbb{F}_q)$ . Put  $G_\mu(q) := \#\mathfrak{G}_\mu(\mathbb{F}_q)$  and let  $A_\mu(q)$  be the  $A$ -polynomial of the quiver  $\Gamma_\mu$  for the dimension vector  $\mathbf{v}_\mu$ .

**Theorem 3.2.3.** *We have*

$$\text{Log} \left( \sum_{\mu \in \mathcal{P}^k} G_\mu(q) m_\mu \right) = \sum_{\mu \in \mathcal{P}^k - \{0\}} A_\mu(q) m_\mu.$$

*Proof.* In Proposition 3.2.2 make the change of variables

$$X_0 := x_{1,1} \cdots x_{k,1}, \quad X_{[i,j]} := x_{i,j}^{-1} x_{i,j+1}, \quad i = 1, 2, \dots, k, \quad j = 1, 2, \dots$$

Since the terms on both sides are invariant under permutation of the entries  $v_{[i,1]}, v_{[i,2]}, \dots$  of  $\mathbf{v}$  we can collect all terms that yield the same multipartition  $\mu$ . The resulting sum of  $X^\mathbf{v}$  gives the monomial symmetric function  $m_\mu(x)$ . □

*Remark 3.2.4.* Since  $A_\mu(q) \in \mathbb{Z}[q]$ , it follows from Theorem 3.2.3 that  $G(q) \in \mathbb{Z}[q]$ .

Recall that  $\mathbb{F}$  denotes an algebraic closure of  $\mathbb{F}_q$  and  $f : \mathbb{F} \rightarrow \mathbb{F}, x \mapsto x^q$  is the Frobenius endomorphism.

**Proposition 3.2.5.** *We have*

$$\log \left( \sum_{\mu} G_{\mu}(q) m_{\mu} \right) = \sum_{d=1}^{\infty} \phi_d(q) \cdot \log \left( \Omega(\mathbf{x}_1^d, \dots, \mathbf{x}_k^d, 0, q^{d/2}) \right)$$

where  $\phi_n(q) = \frac{1}{n} \sum_{d|n} \mu(d)(q^{n/d} - 1)$  is the number of  $\langle f \rangle$ -orbits of  $\mathbb{F}^{\times} := \mathbb{F} - \{0\}$  of size  $n$ .

*Proof.* If  $X$  is a finite set on which a finite group  $H$  acts, recall Burnside's formula which says that

$$\#X/H = \frac{1}{|H|} \sum_{h \in H} \#\{x \in X \mid h \cdot x = x\}.$$

Denote by  $C_n$  the set of conjugacy classes of  $\mathrm{GL}_n(\mathbb{F}_q)$ . Applying Burnside's formula to  $\mathfrak{G}_{\mu}(\mathbb{F}_q)$ , with  $\mu \in (\mathcal{P}_n)^k$ , we find that

$$\begin{aligned} G_{\mu}(q) &= |\mathrm{GL}_n(\mathbb{F}_q)|^{-1} \sum_{g \in \mathrm{GL}_n(\mathbb{F}_q)} \Lambda(g) \prod_{i=1}^k \#\{X \in \mathcal{F}_{\mu^i} \mid g \cdot X = X\} \\ &= |\mathrm{GL}_n(\mathbb{F}_q)|^{-1} \sum_{g \in \mathrm{GL}_n(\mathbb{F}_q)} \Lambda(g) \prod_{i=1}^k R_{L_{\mu^i}}^G(1)(g) \\ &= \sum_{O \in C_n} \frac{\Lambda(O)}{|Z_O|} \prod_{i=1}^k R_{L_{\mu^i}}^G(1)(O) \end{aligned}$$

For a conjugacy class  $O$  of  $\mathrm{GL}_n(\mathbb{F}_q)$ , let  $\omega(O)$  denotes its type. By Formula (2.1.9), we have

$$\frac{\Lambda(O)}{|Z_O|} = \mathcal{H}_{\omega(O)}(0, \sqrt{q}).$$

By Corollary 2.2.3, we deduce that

$$\sum_{\mu} G_{\mu}(q) m_{\mu} = \sum_{O \in C} \mathcal{H}_{\omega(O)}(0, \sqrt{q}) \prod_{i=1}^k \tilde{H}_{\omega(O)}(\mathbf{x}_i, q)$$

where  $C := \bigcup_{n \geq 1} C_n$ .

We denote by  $\mathbf{F}^{\times}$  the set of  $\langle f \rangle$ -orbits of  $\mathbb{F}^{\times}$ . There is a natural bijection from the set  $C$  to the set of all maps  $\mathbf{F}^{\times} \rightarrow \mathcal{P}$  with finite support [24, IV, 2]. If  $C \in C$  corresponds to  $\alpha : \mathbf{F}^{\times} \rightarrow \mathcal{P}$ , then we may enumerate the elements of  $\{s \in \mathbf{F}^{\times} \mid \alpha(s) \neq 0\}$  as  $c_1, \dots, c_r$  such that  $\omega(\alpha) := (d(c_1), \alpha(c_1)) \cdots (d(c_r), \alpha(c_r))$ , where  $d(c)$  denotes the size of  $c$ , is the type  $\omega(C)$ .

We have

$$\begin{aligned} \sum_{\mu} G_{\mu}(q) m_{\mu} &= \sum_{\alpha \in \mathcal{P}^{\mathbf{F}^{\times}}} \mathcal{H}_{\omega(\alpha)}(0, \sqrt{q}) \prod_{i=1}^k \tilde{H}_{\omega(\alpha)}(\mathbf{x}_i, q) \\ &= \prod_{c \in \mathbf{F}^{\times}} \Omega(\mathbf{x}_1^{d(c)}, \dots, \mathbf{x}_k^{d(c)}; 0, q^{d(c)/2}) \\ &= \prod_{d=1}^{\infty} \Omega(\mathbf{x}_1^d, \dots, \mathbf{x}_k^d; 0, q^{d/2})^{\phi_d(q)} \end{aligned}$$

□

*Remark 3.2.6.* The second formula displayed in the proof of Proposition 3.2.5 shows that

$$G_\mu(q) = \langle \Lambda \otimes R_\mu(1), 1 \rangle$$

where  $R_\mu(1) := R_{L_{\mu^1}}^G(1) \otimes \cdots \otimes R_{L_{\mu^k}}^G(1)$ .

**Theorem 3.2.7.** *We have*

$$A_\mu(q) = \mathbb{H}_\mu(0, \sqrt{q}).$$

*Proof.* From Formula (2.1.15) we have

$$\sum_{\mu} \mathbb{H}_\mu(0, \sqrt{q}) m_\mu = (q-1) \text{Log}(\Omega(0, \sqrt{q})).$$

We thus need to see that

$$\sum_{\mu} A_\mu(q) m_\mu = (q-1) \text{Log}(\Omega(0, \sqrt{q})). \quad (3.2.1)$$

From Theorem 3.2.3 we are reduced to prove that

$$\text{Log} \left( \sum_{\mu} G_\mu(q) m_\mu \right) = (q-1) \text{Log}(\Omega(0, \sqrt{q})).$$

But this follows from Lemma 2.1.2 and Proposition 3.2.5. □

### 3.3 Another formula for the A-polynomial

When the dimension vector  $\mathbf{v}_\mu$  is indivisible, it is known by Crawley-Boevey and van den Bergh [1] that the polynomial  $A_\mu(q)$  equals (up to some power of  $q$ ) to the polynomial which counts the number of points of some quiver variety over  $\mathbb{F}_q$ .

Here we prove some relation between  $A_\mu(q)$  and some variety which is closely related to quiver varieties. This relation holds for any  $\mu$  (in particular  $\mathbf{v}_\mu$  can be divisible).

We continue to use the notation  $G, P_\lambda, L_\lambda, U_\lambda, \mathcal{F}_\lambda$  of §2.2 and the notation  $\mathfrak{g}, \mathfrak{p}_\lambda, \mathfrak{l}_\lambda, \mathfrak{u}_\lambda$  of §2.2.2.

For a partition  $\lambda$  of  $n$ , define

$$\mathbb{X}_\lambda := \left\{ (X, gP_\lambda) \in \mathfrak{g} \times (G/P_\lambda) \mid g^{-1}Xg \in \mathfrak{u}_\lambda \right\}$$

It is well-known that the image of the projection  $p : \mathbb{X}_\lambda(\mathbb{F}) \rightarrow \mathfrak{g}(\mathbb{F}), (X, gP_\lambda) \mapsto X$  is the Zariski closure  $\overline{\mathcal{O}}_{\lambda'}$  of the nilpotent adjoint orbit  $\mathcal{O}_{\lambda'}$  of  $\mathfrak{gl}_n(\mathbb{F})$  whose Jordan form is given by  $\lambda'$ , and that  $p$  is a desingularization.

Put

$$\mathbb{V}_\mu := \left\{ (a_1, b_1, \dots, a_g, b_g, (X_1, g_1 P_{\mu^1}), \dots, (X_k, g_k P_{\mu^k})) \in \mathfrak{g}^{2g} \times \mathbb{X}_{\mu^1} \times \cdots \times \mathbb{X}_{\mu^k} \mid \sum_i [a_i, b_i] + \sum_j X_j = 0 \right\}$$

where  $[a, b] = ab - ba$ .

Define  $\Lambda^\sim : \mathfrak{g} \rightarrow \mathbb{C}, z \mapsto q^{gn^2} \Lambda(z)$ . By [10, Proposition 3.2.2] we know that

$$\Lambda^\sim = \mathcal{F}^{\mathfrak{g}}(F)$$

where for  $z \in \mathfrak{g}$ ,

$$F(z) := \# \left\{ (a_1, b_1, \dots, a_g, b_g) \in \mathfrak{g}^{2g} \mid \sum_i [a_i, b_i] = z \right\}.$$

By Remark 2.2.11, the functions  $\Lambda^\sim$  and  $\mathfrak{R}_{\mathfrak{l}_\lambda}^{\mathfrak{g}} := q^{\frac{1}{2}(n^2 - \sum_i \lambda_i^2)} R_{\mathfrak{l}_\lambda}^{\mathfrak{g}}$  are characters of  $\mathfrak{g}$ . Put

$$\mathfrak{R}_\mu(1) := \mathfrak{R}_{\mathfrak{l}_{\mu^1}}^{\mathfrak{g}}(1) \otimes \cdots \otimes \mathfrak{R}_{\mathfrak{l}_{\mu^k}}^{\mathfrak{g}}(1).$$

For two functions  $f, g : \mathfrak{g} \rightarrow \mathbb{C}$ , define their inner product as

$$\langle f, g \rangle = |\mathfrak{g}|^{-1} \sum_{X \in \mathfrak{g}} f(X) \overline{g(X)}.$$

**Proposition 3.3.1.** *We have*

$$|\mathbb{V}_\mu| = \langle \Lambda^\sim \otimes \mathfrak{R}_\mu(1), 1 \rangle.$$

*Proof.* Notice that

$$|\mathbb{V}_\mu| = \left( F * Q_{\mathfrak{l}_{\mu^1}}^{\mathfrak{g}} * \cdots * Q_{\mathfrak{l}_{\mu^k}}^{\mathfrak{g}} \right)(0).$$

Hence the result follows from Proposition 2.2.8 and Proposition 2.2.10.  $\square$

The proposition shows that  $|\mathbb{V}_\mu|$  is a rational function in  $q$  which is an integer for infinitely many values of  $q$ . Hence  $|\mathbb{V}_\mu|$  is a polynomial in  $q$  with integer coefficients.

Consider

$$V_\mu(q) := \frac{|\mathbb{V}_\mu|}{|G|}.$$

Recall that  $d_\mu = n^2(2g - 2 + k) - \sum_{i,j} (\mu_j^i)^2 + 2$ .

**Theorem 3.3.2.** *We have*

$$\text{Log} \left( \sum_\mu q^{-\frac{1}{2}(d_\mu - 2)} V_\mu(q) m_\mu \right) = \frac{q}{q-1} \sum_\mu A_\mu(q) m_\mu.$$

By Lemma 2.1.2 and Formula (3.2.1) we are reduced to prove the following.

**Proposition 3.3.3.** *We have*

$$\log \left( \sum_\mu q^{-\frac{1}{2}(d_\mu - 2)} V_\mu(q) m_\mu \right) = \sum_{d=1}^{\infty} \varphi_d(q) \cdot \log \left( \Omega(\mathbf{x}_1^d, \dots, \mathbf{x}_k^d, 0, q^{d/2}) \right)$$

where  $\varphi_n(q) = \frac{1}{n} \sum_{d|n} \mu(d) q^{n/d}$  is the number of  $\langle f \rangle$ -orbits of  $\mathbb{F}$  of size  $n$ .

*Proof.* By Proposition 3.3.1, we have

$$V_\mu(q) = \frac{q^{-n^2 + \frac{1}{2}(kn^2 - \sum_{i,j} (\mu_j^i)^2)}}{|G|} \sum_{x \in \mathfrak{g}} \Lambda^\sim(x) R_{\mathfrak{l}_{\mu^1}}^{\mathfrak{g}}(1)(x) \cdots R_{\mathfrak{l}_{\mu^k}}^{\mathfrak{g}}(1)(x).$$

By Remark 2.2.9 and Corollary 2.2.3, we see that  $R_{\mathfrak{l}_\lambda}^{\mathfrak{g}}(1)(x) = \langle \tilde{H}_\omega(\mathbf{x}; q), h_\lambda(\mathbf{x}) \rangle$  when the  $G$ -orbit of  $x$  is of type  $\omega$ .

We now proceed exactly as in the proof of Proposition 3.2.5 to prove our formula.  $\square$

### 3.4 Applications to the character theory of finite general linear groups

The following theorem (which is a consequence of Theorem 3.2.7 and Theorem 2.2.6) expresses certain fusion rules in the character ring of  $\mathrm{GL}_n(\mathbb{F}_q)$  in terms of absolutely indecomposable representations of comet shaped quivers.

**Theorem 3.4.1.** *We have*

$$\langle \Lambda \otimes R_\mu, 1 \rangle = A_\mu(q).$$

From Theorem 3.4.1 and Theorem 3.1.1 we have the following result.

**Corollary 3.4.2.**  *$\langle \Lambda \otimes R_\mu, 1 \rangle \neq 0$  if and only if  $\mathbf{v}_\mu \in \Phi(\Gamma_\mu)^+$ . Moreover  $\langle \Lambda \otimes R_\mu, 1 \rangle = 1$  if and only if  $\mathbf{v}_\mu$  is a real root.*

*Remark 3.4.3.* We will see in §5.2 that  $\mathbf{v}_\mu$  is always an imaginary root when  $g \geq 1$ , hence the second assertion concerns only the case  $g = 0$  (i.e.  $\Lambda = 1$ ).

A proof of Theorem 3.4.1 for  $\mathbf{v}_\mu$  is indivisible is given in [10] by expressing  $\langle \Lambda \otimes R_\mu, 1 \rangle$  as the Poincaré polynomial of a comet-shaped quiver variety. This quiver variety exists only when  $\mathbf{v}_\mu$  is indivisible.

In [13] the authors discuss some results of Mattig who showed that the multiplicities  $\langle \mathcal{X}_1 \otimes \mathcal{X}_2, \mathcal{X}_3 \rangle$ , with  $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$  unipotent characters of  $\mathrm{GL}_n$ , are polynomials in  $q$  with rational coefficients. Using calculation with CHEVIE, he also observed when  $n \leq 8$  that the coefficients of  $\langle \mathcal{X}_1 \otimes \mathcal{X}_2, \mathcal{X}_3 \rangle$  are non-negative integers. The following proposition confirms part of this prediction.

**Proposition 3.4.4.** *For a  $k$ -tuple of partitions  $\mu = (\mu^1, \dots, \mu^k)$  of  $n$  there exists a polynomial  $U_\mu \in \mathbb{Z}[T]$  such that  $\langle \Lambda \otimes \mathcal{U}_{\mu^1} \otimes \dots \otimes \mathcal{U}_{\mu^k}, 1 \rangle = U_\mu(q)$ .*

*Proof.* Since the complete symmetric functions  $\{h_\lambda(\mathbf{x})\}_\lambda$  forms a  $\mathbb{Z}$ -basis of the ring  $\Lambda(\mathbf{x})$ , for any partitions  $\mu, \lambda$ , there exist integers  $a_{\mu\lambda}$  such that

$$s_\mu(\mathbf{x}) = \sum_\lambda a_{\mu\lambda} h_\lambda(\mathbf{x}).$$

The matrix  $(a_{\mu\lambda})_{\mu,\lambda}$  is the transpose inverse of the matrix  $K = (K_{\mu\lambda})_{\mu,\lambda}$  of Kostka numbers.

If  $\mathcal{U}_\mu$  is the unipotent characters of  $\mathrm{GL}_n$  corresponding to  $\mu$ , by Theorem 2.2.2 we have

$$\mathcal{U}_\mu(C) = \langle \tilde{H}_\omega(\mathbf{x}; q), s_\mu(\mathbf{x}) \rangle$$

where  $C$  is a conjugacy class of type  $\omega$ . Hence by Corollary 2.2.3, we deduce that

$$\mathcal{U}_\mu = \sum_\lambda a_{\mu\lambda} R_{L_\lambda}^G(1).$$

Hence  $\langle \Lambda \otimes \mathcal{U}_1 \otimes \dots \otimes \mathcal{U}_k, 1 \rangle$  is a  $\mathbb{Z}$ -linear combination of multiplicities of the form

$$\langle \Lambda \otimes R_{L_{\lambda^1}}^G(1) \otimes \dots \otimes R_{L_{\lambda^k}}^G(1), 1 \rangle$$

and so by Remark 3.2.6, it is a  $\mathbb{Z}$ -linear combination of polynomials of the form  $G_\mu(q)$  which have integer coefficients (see Remark 3.2.4).  $\square$

## 4 Example: Hilbert Scheme of $n$ points on $\mathbb{C}^\times \times \mathbb{C}^\times$

Throughout this section we will have  $g = k = 1$  and  $\mu$  will be either the partition  $(n)$  or  $(n-1, 1)$ .

In this section we illustrate our conjectures and formulas in these cases.

#### 4.1 Hilbert schemes: Review

For a nonsingular complex surface  $S$  we denote by  $S^{[n]}$  the Hilbert scheme of  $n$  points in  $S$ . Recall that  $S^{[n]}$  is nonsingular and has dimension  $2n$ .

We denote by  $Y^{[n]}$  the Hilbert scheme of  $n$  points in  $\mathbb{C}^2$ .

Recall (see for instance [27, §5.2]) that  $h_c^i(Y^{[n]}) = 0$  unless  $i$  is even and that the compactly supported Poincaré polynomial  $P_c(Y^{[n]}; q) := \sum_i h_c^{2i}(Y^{[n]})q^i$  is given by the following explicit formula

$$\sum_{n \geq 0} P_c(Y^{[n]}; q) T^n = \prod_{m \geq 1} \frac{1}{1 - q^{m+1} T^m}. \quad (4.1.1)$$

which is equivalent to

$$\text{Log} \left( \sum_{n \geq 0} q^{-n} \cdot P_c(Y^{[n]}; q) T^n \right) = \sum_{n \geq 1} q T^n. \quad (4.1.2)$$

For  $n \geq 2$ , consider the partition  $\mu = (n-1, 1)$  of  $n$  and let  $C$  be a semisimple adjoint orbit of  $\mathfrak{gl}_n(\mathbb{C})$  with characteristic polynomial of the form  $(-1)^n(x - \alpha)^{n-1}(x - \beta)$  with  $\beta = -(n-1)\alpha$  and  $\alpha \neq 0$ . Consider the variety

$$\mathcal{V}_{(n-1,1)} = \{(a, b, X) \in (\mathfrak{gl}_n)^2 \times C \mid [a, b] + X = 0\}.$$

The group  $\text{GL}_n$  acts on  $\mathcal{V}_{(n-1,1)}$  diagonally by conjugating the coordinates. This action induces a free action of  $\text{PGL}_n$  on  $\mathcal{V}_{(n-1,1)}$  and we put

$$\mathcal{Q}_{(n-1,1)} := \mathcal{V}_{(n-1,1)} // \text{PGL}_n = \text{Spec} \left( \mathbb{C}[\mathcal{V}_{(n-1,1)}]^{\text{PGL}_n} \right).$$

The variety  $\mathcal{Q}_{(n-1,1)}$  is known to be nonsingular of dimension  $2n$  (see for instance [10, §2.2] and the references therein).

We have the following well-known theorem.

**Theorem 4.1.1.** *The two varieties  $\mathcal{Q}_{(n-1,1)}$  and  $Y^{[n]}$  have isomorphic cohomology supporting pure mixed Hodge structures.*

*Proof.* By [10, Appendix B] it is enough to prove that there is a smooth morphism  $f : \mathfrak{M} \rightarrow \mathbb{C}$  which satisfies the two following properties:

- (1) There exists an action of  $\mathbb{C}^\times$  on  $\mathfrak{M}$  such that the fixed point set  $\mathfrak{M}^{\mathbb{C}^\times}$  is complete and for all  $x \in X$  the limit  $\lim_{\lambda \rightarrow 0} \lambda x$  exists.
- (2)  $\mathcal{Q}_{(n-1,1)} = f^{-1}(\lambda)$  and  $Y^{[n]} = f^{-1}(0)$ .

Denote by  $\mathbf{v}$  the dimension vector of  $\Gamma_{(n-1,1)}$  which has coordinate  $n$  on the central vertex (i.e., the vertex supporting the loop) and 1 on the other vertex. It is well-known (see Nakajima [27]) that  $Y^{[n]}$  can be identified with the quiver variety  $\mathfrak{M}_{0,\theta}(\mathbf{v})$  where  $\theta$  is the stability parameter with coordinate  $-1$  on the central vertex and  $n$  on the other vertex. If we let  $\xi$  be the parameter with coordinate  $-\alpha$  at the central vertex and  $\alpha - \beta$  at the other vertex, then the variety  $\mathcal{Q}_{(n-1,1)}$  is isomorphic to the quiver variety  $\mathfrak{M}_{\xi,\theta}(\mathbf{v})$  (see for instance [10] and the references therein). Now we can define as in [10, §2.2] a map  $f : \mathfrak{M} \rightarrow \mathbb{C}$  such that  $f^{-1}(0) = \mathfrak{M}_{0,\theta}(\mathbf{v})$  and  $f^{-1}(\lambda) = \mathfrak{M}_{\xi,\theta}(\mathbf{v})$  and which satisfies the required properties.  $\square$

**Proposition 4.1.2.** *We have*

$$P_c(Y^{[n]}; q) = q^n \cdot A_{(n-1,1)}(q).$$

*Proof.* We have  $P_c(\mathcal{Q}_{(n-1,1)}; q) = q^n \cdot \mathbb{H}_{(n-1,1)}(0, \sqrt{q})$  by [10, Theorem 1.3.1] and so by Theorem 3.2.7 we see that  $P_c(\mathcal{Q}_{(n-1,1)}; q) = q^n \cdot A_{(n-1,1)}(q)$ . Hence the result follows from Theorem 4.1.1.  $\square$

Now put  $X := \mathbb{C}^* \times \mathbb{C}^*$ . Unlike  $Y^{[n]}$ , the mixed Hodge structure on  $X^{[n]}$  is not pure. By Göttsche and Soergel [9] we have the following result.

**Theorem 4.1.3.** *We have  $h_c^{i,j,k}(X^{[n]}) = 0$  unless  $i = j$  and*

$$1 + \sum_{n \geq 1} H_c(X^{[n]}; q, t) T^n = \prod_{n \geq 1} \frac{(1 + t^{2n+1} q^n T^n)^2}{(1 - q^{n-1} t^{2n} T^n)(1 - t^{2n+2} q^{n+1} T^n)} \quad (4.1.3)$$

with  $H_c(X^{[n]}; q, t) := \sum_{i,k} h_c^{i,i,k}(X^{[n]}) q^i t^k$ .

Define  $\mathbb{H}^{[n]}(z, w)$  such that

$$H_c(X^{[n]}; q, t) = (t \sqrt{q})^{2n} \mathbb{H}^{[n]} \left( -t \sqrt{q}, \frac{1}{\sqrt{q}} \right).$$

Then Formula (4.1.3) reads

$$\sum_{n \geq 0} \mathbb{H}^{[n]}(z, w) T^n = \prod_{n \geq 1} \frac{(1 - zw T^n)^2}{(1 - z^2 T^n)(1 - w^2 T^n)}, \quad (4.1.4)$$

with the convention that  $\mathbb{H}^{[0]}(z, w) = 1$ . Hence we may re-write Formula (4.1.3) as

$$\text{Log} \left( \sum_{n \geq 0} \mathbb{H}^{[n]}(z, w) T^n \right) = \sum_{n \geq 1} (z - w)^2 T^n. \quad (4.1.5)$$

Specializing Formula (4.1.5) with  $(z, w) \mapsto (0, \sqrt{q})$  we see from Formula (4.1.2) that

$$P_c(Y^{[n]}; q) = q^n \cdot \mathbb{H}^{[n]}(0, \sqrt{q}). \quad (4.1.6)$$

We thus have the following result.

**Proposition 4.1.4.** *We have*

$$PH_c(X^{[n]}; T) = P_c(Y^{[n]}; T).$$

where  $PH_c(X^{[n]}; T) := \sum_i h_c^{i,i,2i}(X^{[n]}) T^i$  is the Poincaré polynomial of the pure part of the cohomology of  $X^{[n]}$ .

## 4.2 A conjecture

The aim of this section is to discuss the following conjecture.

**Conjecture 4.2.1.** *We have*

$$\mathbb{H}_{(n-1,1)}(z, w) = \mathbb{H}^{[n]}(z, w). \quad (4.2.1)$$

Modulo the conjectural formula (1.1.1), Formula (4.2.1) says that the two mixed Hodge polynomials  $H_c(X^{[n]}; q, t)$  and  $H_c(\mathcal{M}_{(n-1,1)}; q, t)$  agree. This would be a multiplicative analogue of Theorem 4.1.1. Unfortunately the proof of Theorem 4.1.1 does not work in the multiplicative case. This is because the natural family  $g : \mathfrak{X} \rightarrow \mathbb{C}$  with  $X^{[n]} = g^{-1}(0)$  and  $\mathcal{M}_{(n-1,1)} = g^{-1}(\lambda)$  for  $0 \neq \lambda \in \mathbb{C}$  does not support a  $\mathbb{C}^\times$ -action with a projective fixed point set and so [10, Appendix B] does not apply.

One can still attempt to prove that the restriction map  $H^*(\mathfrak{X}; \mathbb{Q}) \rightarrow H^*(g^{-1}(\lambda); \mathbb{Q})$  is an isomorphism for every fibre over  $\lambda \in \mathbb{C}$  by using a family version of the non-Abelian Hodge theory as developed in the tamely ramified case in [28]. In other words one would construct a family  $g_{\text{Dol}} : \mathfrak{X}_{\text{Dol}} \rightarrow \mathbb{C}$  such that

$g_{\text{Dol}}^{-1}(0)$  would be isomorphic with the moduli space of parabolic Higgs bundles on an elliptic curve  $C$  with one puncture and flag type  $(n-1, 1)$  and meromorphic Higgs field with a nilpotent residue at the puncture, and  $g_{\text{Dol}}^{-1}(\lambda)$  for  $\lambda \neq 0$  would be isomorphic with parabolic Higgs bundles on  $C$  with one puncture and semisimple residue at the puncture of type  $(n-1, 1)$ . In this family one should have a  $\mathbb{C}^\times$  action satisfying the assumptions of [10, Appendix B] and so could conclude that  $H^*(\mathfrak{X}_{\text{Dol}}; \mathbb{Q}) \rightarrow H^*(g_{\text{Dol}}^{-1}(\lambda); \mathbb{Q})$  is an isomorphism for every fibre over  $\lambda \in \mathbb{C}$ . Then a family version of non-Abelian Hodge theory in the tamely ramified case would yield that the two families  $\mathfrak{X}_{\text{Dol}}$  and  $\mathfrak{X}$  are diffeomorphic, and so one could conclude the desired isomorphism  $H^*(X^{[n]}; \mathbb{Q}) \cong H^*(\mathcal{M}_{(n-1,1)})$  preserving mixed Hodge structures. However a family version of the non-Abelian Hodge theory in the tamely ramified case (which was initiated in [28]) is not available in the literature.

**Proposition 4.2.2.** *Conjecture 4.2.1 is true under the specialization  $z = 0, w = \sqrt{q}$ .*

*Proof.* The left hand side specializes to  $A_{(n-1,1)}(q)$  by Theorem 3.2.7, which by (4.1.5) and Proposition 4.1.2 agrees with the right hand side.  $\square$

The *Young diagram* of a partition  $\lambda = (\lambda_1, \lambda_2, \dots)$  is defined as the set of points  $(i, j) \in \mathbb{Z}^2$  such that  $1 \leq j \leq \lambda_i$ . We adopt the convention that the coordinate  $i$  of  $(i, j)$  increases as one goes down and the second coordinate  $j$  increases as one goes to the right.

For  $\lambda \neq 0$ , we define  $\phi_\lambda(z, w) := \sum_{(i,j) \in \lambda} z^{i-1} w^{j-1}$ , and for  $\lambda = 0$ , we put  $\phi_\lambda(z, w) = 0$ . Define

$$\begin{aligned} A_1(z, w; T) &:= \sum_{\lambda} \mathcal{H}_\lambda(z, w) \phi_\lambda(z^2, w^2) T^{|\lambda|}, \\ A_0(z, w; T) &:= \sum_{\lambda} \mathcal{H}_\lambda(z, w) T^{|\lambda|}. \end{aligned}$$

**Proposition 4.2.3.** *We have*

$$\sum_{n \geq 1} \mathbb{H}_{(n-1,1)}(z, w) T^n = (z^2 - 1)(1 - w^2) \frac{A_1(z, w; T)}{A_0(z, w; T)}.$$

*Proof.* The coefficient of the monomial symmetric function  $m_{(n-1,1)}(\mathbf{x})$  in a symmetric function in  $\Lambda(\mathbf{x})$  of homogeneous degree  $n$  is the coefficient of  $u$  when specializing the variables  $\mathbf{x} = \{x_1, x_2, \dots\}$  to  $\{1, u, 0, 0, \dots\}$ . Hence, the generating series  $\sum_{n \geq 1} \mathbb{H}_{(n-1,1)}(z, w) T^n$  is the coefficient of  $u$  in

$$(z^2 - 1)(1 - w^2) \text{Log} \left( \sum_{\lambda} \mathcal{H}_\lambda(z, w) \tilde{H}_\lambda(1, u, 0, 0, \dots; z^2, w^2) T^{|\lambda|} \right).$$

We know that

$$\tilde{H}_\lambda(\mathbf{x}; z, w) = \sum_{\rho} \tilde{K}_{\rho\lambda}(z, w) s_{\rho}(\mathbf{x}),$$

and  $s_{\rho}(\mathbf{x}) = \sum_{\mu \leq \rho} K_{\rho\mu} m_{\mu}(\mathbf{x})$  where  $K_{\rho\mu}$  are the Kostka numbers. We have

$$\begin{aligned} s_{(n)}(1, u, 0, 0, \dots) &= 1 + u + O(u^2) \\ s_{(n-1,1)}(1, u, 0, 0, \dots) &= u + O(u^2) \end{aligned}$$

and

$$s_{\rho}(1, u, 0, 0, \dots) = O(u^2)$$

for any other partition  $\rho$ . Hence,

$$\tilde{H}_\lambda(1, u, 0, 0, \dots; z, w) = \tilde{K}_{(n)\lambda}(z, w)(1 + u) + \tilde{K}_{(n-1,1)\lambda}(z, w)u + O(u^2).$$



From Macdonald [24, p. 362] we obtain  $\tilde{K}_{(n)\lambda}(a, b) = 1$  and  $\tilde{K}_{(n-1,1)\lambda}(a, b) = \phi_\lambda(a, b) - 1$ . Hence, finally,

$$\tilde{H}_\lambda(1, u, 0, 0, \dots; z, w) = 1 + \phi_\lambda(z, w)u + O(u^2). \quad (4.2.2)$$

It follows that  $(z^2 - 1)^{-1}(1 - w^2)^{-1} \sum_{n \geq 1} \mathbb{H}_{(n-1,1)}(z, w)T^n$  equals the coefficient of  $u$  in

$$\text{Log} \left( \sum_{\lambda} \mathcal{H}_\lambda(z, w) \left( 1 + \phi_\lambda(z^2, w^2)u + O(u^2) \right) T^{|\lambda|} \right) = \text{Log} \left( A_0(T) + A_1(T)u + O(u^2) \right).$$

The claim follows from the general fact

$$\text{Log} \left( A_0(T) + A_1(T)u + O(u^2) \right) = \text{Log} A_0(T) + \frac{A_1(T)}{A_0(T)}u + O(u^2).$$

□

Combining Proposition 4.2.3 with (4.1.4) we obtain the following.

**Corollary 4.2.4.** *Conjecture 4.2.1 is equivalent to the following combinatorial identity*

$$1 + (z^2 - 1)(1 - w^2) \frac{A_1(z, w; T)}{A_0(z, w; T)} = \prod_{n \geq 1} \frac{(1 - zwT^n)^2}{(1 - z^2T^n)(1 - w^2T^n)}. \quad (4.2.3)$$

The main result of this section is the following theorem.

**Theorem 4.2.5.** *Formula (4.2.3) is true under the Euler specialization  $(z, w) \mapsto (\sqrt{q}, 1/\sqrt{q})$ ; namely, we have*

$$\mathbb{H}_{(n-1,1)}(z, z^{-1}) = \mathbb{H}^{[n]}(z, z^{-1}). \quad (4.2.4)$$

*Equivalently, the two varieties  $\mathcal{M}_{(n-1,1)}$  and  $X^{[n]}$  have the same E-polynomial.*

*Proof.* Consider the generating function

$$F := (1 - z)(1 - w) \sum_{\lambda} \phi_\lambda(z, w) T^{|\lambda|}.$$

It is straightforward to see that for  $\lambda \neq 0$  we have

$$\begin{aligned} (1 - z)(1 - w)\phi_\lambda(z, w) &= 1 + \sum_{i=1}^{l(\lambda)} (w^i - w^{i-1})z^{\lambda_i} - w^{l(\lambda)} \\ &= 1 + \sum_{i \geq 1} (w^i - w^{i-1})z^{\lambda_i}. \end{aligned}$$

Interchanging summations we find

$$F = \sum_{i \geq 1} (w^i - w^{i-1}) \sum_{\lambda \neq 0} z^{\lambda_i} T^{|\lambda|} + \sum_{\lambda \neq 0} T^{|\lambda|}.$$

To compute the sum over  $\lambda$  for a fixed  $i$  we break the partitions as follows:

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{i-1} \geq \underbrace{\lambda_i \geq \lambda_{i+1} \geq \dots}_{\rho}$$

and we put

$$\begin{aligned} \rho &:= (\lambda_i, \lambda_{i+1}, \dots) \\ \mu &:= (\lambda_1 - \lambda_i, \lambda_2 - \lambda_i, \dots, \lambda_{i-1} - \lambda_i) \end{aligned}$$

Notice that  $\mu'_1 = l(\mu) < i$ ,  $\rho_1 = l(\rho') = \lambda_i$  and  $|\lambda| = |\mu| + |\rho| + l(\rho')(i-1)$ .

We then have

$$\sum_{\lambda} z^{\lambda_i} T^{|\lambda|} = \sum_{\mu_1 < i} T^{|\mu|} \sum_{\rho} z^{l(\rho)} T^{|\rho| + (i-1)l(\rho)}$$

(changing  $\rho$  to  $\rho'$  and  $\mu$  to  $\mu'$ ). Each sum can be written as an infinite product, namely

$$\sum_{\lambda} z^{\lambda_i} T^{|\lambda|} = \prod_{k=1}^{i-1} (1 - T^k)^{-1} \prod_{n \geq 1} (1 - zT^{n+i-1})^{-1}.$$

So

$$\begin{aligned} F &= \sum_{\lambda \neq 0} T^{|\lambda|} + \sum_{i \geq 1} (w^i - w^{i-1}) \left( \prod_{k=1}^{i-1} (1 - T^k)^{-1} \prod_{n \geq 1} (1 - zT^{n+i-1})^{-1} - 1 \right) \\ &= \sum_{\lambda \neq 0} T^{|\lambda|} + \prod_{n \geq 1} (1 - zT^n)^{-1} \sum_{i \geq 1} (w^i - w^{i-1}) \prod_{k=1}^{i-1} \frac{(1 - zT^k)}{(1 - T^k)} - \sum_{i \geq 1} (w^i - w^{i-1}). \end{aligned}$$

The last sum telescopes to 1 and we find

$$F = \sum_{\lambda} T^{|\lambda|} + \prod_{n \geq 1} (1 - zT^n)^{-1} (w - 1) \sum_{i \geq 1} w^{i-1} \prod_{k=1}^{i-1} \frac{(1 - zT^k)}{(1 - T^k)}. \quad (4.2.5)$$

By the Cauchy  $q$ -binomial theorem the sum equals

$$\frac{1}{(1 - w)} \prod_{n \geq 1} \frac{(1 - wzT^n)}{(1 - wT^n)}.$$

Also

$$\sum_{\lambda} T^{|\lambda|} = \prod_{n \geq 1} (1 - T^n)^{-1}.$$

If we divide Formula (4.2.5) by this we finally get

$$1 - (1 - z)(1 - w) \prod_{n \geq 1} (1 - T^n) \sum_{\lambda} \phi_{\lambda}(z, w) T^{|\lambda|} = \prod_{n \geq 1} \frac{(1 - wzT^n)(1 - T^n)}{(1 - zT^n)(1 - wT^n)}.$$

Putting now  $(z, w) = (q, 1/q)$  we find that

$$1 - (1 - q)(1 - 1/q) \prod_{n \geq 1} (1 - T^n) \sum_{\lambda} \phi_{\lambda}(q, 1/q) T^{|\lambda|} = \prod_{n \geq 1} \frac{(1 - T^n)^2}{(1 - qT^n)(1 - q^{-1}T^n)}. \quad (4.2.6)$$

From Formula (2.1.10) we have  $\mathcal{H}_{\lambda}(\sqrt{q}, 1/\sqrt{q}) = 1$  and so

$$\begin{aligned} A_1 \left( \sqrt{q}, \frac{1}{\sqrt{q}}; T \right) &= \sum_{\lambda} \phi_{\lambda} \left( q, \frac{1}{q} \right) T^{|\lambda|} \\ A_0 \left( \sqrt{q}, \frac{1}{\sqrt{q}}; T \right) &= \sum_{\lambda} T^{|\lambda|} = \prod_{n \geq 1} (1 - T^n)^{-1} \end{aligned}$$

Hence, under the specialization  $(z, w) \mapsto (\sqrt{q}, 1/\sqrt{q})$ , the left hand side of Formula (4.2.3) agrees with the left hand side of Formula (4.2.6).

Finally, it is straightforward to see that if we put  $(z, w) = (\sqrt{q}, 1/\sqrt{q})$ , then the right hand side of Formula (4.2.3) agrees with the right hand side of Formula (4.2.6), hence the theorem.  $\square$

### 4.3 Connection with modular forms

For a positive, even integer  $k$  let  $G_k$  be the standard Eisenstein series for  $SL_2(\mathbb{Z})$

$$G_k(T) = \frac{-B_k}{2k} + \sum_{n \geq 1} \sum_{d|n} d^{k-1} T^n, \quad (4.3.1)$$

where  $B_k$  is the  $k$ -th Bernoulli number.

For  $k > 2$  the  $G_k$ 's are modular forms of weight  $k$ ; i.e., they are holomorphic (including at infinity) and satisfy

$$G_k\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k G_k(\tau) \quad (4.3.2)$$

for  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \quad T = e^{2\pi i \tau}, \quad \Im \tau > 0.$

For  $k = 2$  we have a similar transformation up to an additive term.

$$G_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 G_2(\tau) - \frac{c}{4\pi i} (c\tau + d). \quad (4.3.3)$$

The ring  $\mathbb{Q}[G_2, G_4, G_6]$  is called the ring of *quasi-modular* forms (see [17]).

**Theorem 4.3.1.** *We have*

$$1 + \sum_{n \geq 1} \mathbb{H}_{(n-1,1)}(e^{u/2}, e^{-u/2}) T^n = \frac{1}{u} (e^{u/2} - e^{-u/2}) \exp\left(2 \sum_{k \geq 2} G_k(T) \frac{u^k}{k!}\right).$$

*In particular, the coefficient of any power of  $u$  on the left hand side is in the ring of quasi-modular forms.*

*Remark 4.3.2.* The relation between the  $E$ -polynomial of the Hilbert scheme of points on a surface and theta functions goes back to Göttsche [8].

*Proof.* Consider the classical theta function

$$\theta(w) = (1-w) \prod_{n \geq 1} \frac{(1-q^n w)(1-q^n w^{-1})}{(1-q^n)^2}, \quad (4.3.4)$$

with simple zeros at  $q^n$ ,  $n \in \mathbb{Z}$  and functional equations

$$\begin{aligned} \text{i)} \quad & \theta(qw) = -w^{-1} \theta(w) \\ \text{ii)} \quad & \theta(w^{-1}) = -w^{-1} \theta(w) \end{aligned} \quad (4.3.5)$$

We have the following expansion

$$\frac{1}{\theta(w)} = \frac{1}{1-w} + \sum_{\substack{n, m \geq 0 \\ n \not\equiv m \pmod{2}}} (-1)^n q^{\frac{nm}{2}} w^{\frac{m-n-1}{2}} \quad (4.3.6)$$

This is classical but not that well known. For a proof see, for example, [15, Chap. VI, p. 453], where it is deduced from a more general expansion due to Kronecker. Namely,

$$\frac{\theta(uv)}{\theta(u)\theta(v)} = \sum_{m, n \geq 0} q^{mn} u^m v^n - \sum_{m, n \geq 1} q^{mn} u^{-m} v^{-n}.$$

(To see this set  $v = u^{-\frac{1}{2}}$  and use the functional equation (4.3.5) to get

$$\frac{1}{\theta(w)} = \frac{1}{1-w} + \sum_{m,n \geq 1} q^{mn} (w^{m-\frac{1}{2}(n+1)} - w^{m+\frac{1}{2}(n-1)}),$$

which is equivalent to (4.3.6).) It is not hard, as was shown to us by J. Tate, to give a direct proof using (4.3.5).

From (4.3.6) we deduce, switching  $q$  to  $T$  and  $w$  to  $q$ , that

$$\prod_{n \geq 1} \frac{(1-T^n)^2}{(1-qT^n)(1-q^{-1}T^n)} = 1 + \sum_{\substack{r,s > 0 \\ r \not\equiv s \pmod{2}}} (-1)^r T^{\frac{rs}{2}} \left( q^{\frac{s-r-1}{2}} - q^{\frac{2-r+1}{2}} \right) \quad (4.3.7)$$

which combined with Theorem 4.2.5 gives

$$\mathbb{H}_{(n-1,1)} \left( \sqrt{q}, \frac{1}{\sqrt{q}} \right) = \sum_{\substack{r,s=2n \\ r \not\equiv s \pmod{2}}} (-1)^r \left( q^{\frac{s-r-1}{2}} - q^{\frac{2-r+1}{2}} \right) \quad (4.3.8)$$

We compute the logarithm of the left hand side of (4.3.7) and get

$$\sum_{m,n \geq 1} (q^m + q^{-m} - 2) \frac{T^{mn}}{m}$$

Applying  $(q \frac{d}{dq})^k$  and then setting  $q = 1$  we obtain

$$\sum_{m,n \geq 1} (m^k + (-m)^k) \frac{T^{mn}}{m},$$

which vanishes identically if  $k$  is odd. For  $k$  even, it equals

$$2 \sum_{n \geq 1} \sum_{d|n} d^{k-1} T^n.$$

Comparing with (4.3.1) we see that this series equals  $2G_k$ , up to the constant term.

Note that if  $q = e^u$  then

$$q \frac{d}{dq} = \frac{d}{du}, \quad q = 1 \leftrightarrow u = 0.$$

Hence ,

$$\log \left( 1 + \sum_{n \geq 1} \mathbb{H}_{(n-1,1)}(e^{u/2}, e^{-u/2}) T^n \right) = \sum_{\substack{k \geq 2 \\ \text{even}}} \left( 2G_k + \frac{B_k}{k} \right) \frac{u^k}{k!}.$$

On the other hand, it is easy to check that

$$u \exp \left( \sum_{k \geq 2} \frac{B_k}{k} \frac{u^k}{k!} \right) = e^{u/2} - e^{-u/2}$$

( $B_k = 0$  if  $k > 1$  is odd.) This proves the claim.  $\square$

## 5 Connectedness of character varieties

### 5.1 The main result

Let  $\mu$  be a multi-partition  $(\mu^1, \dots, \mu^k)$  of  $n$  and let  $\mathcal{M}_\mu$  be a genus  $g$  generic character variety of type  $\mu$  as in §1.1.

**Theorem 5.1.1.** *The character variety  $\mathcal{M}_\mu$  is connected (if not empty).*

Let us now explain the strategy of the proof.

We first need the following lemma.

**Lemma 5.1.2.** *If  $\mathcal{M}_\mu$  is not empty, its number of connected components equals the constant term in  $E(\mathcal{M}_\mu; q)$ .*

*Proof.* The number of connected components of  $\mathcal{M}_\mu$  is  $\dim H^0(\mathcal{M}_\mu, \mathbb{C})$  which is also equal to the mixed Hodge number  $h^{0,0,0}(\mathcal{M}_\mu)$ .

Poincaré duality implies that

$$h^{i,j,k}(\mathcal{M}_\mu) = h_c^{d_\mu-i, d_\mu-j; 2d_\mu-k}(\mathcal{M}_\mu).$$

From Formula (1.1.3) we thus have

$$E(\mathcal{M}_\mu; q) = \sum_i \left( \sum_k (-1)^k h^{i,i,k}(\mathcal{M}_\mu) \right) q^i,$$

On the other hand the mixed Hodge numbers  $h^{i,j,k}(X)$  of any complex non-singular variety  $X$  are zero if  $(i, j, k) \notin \{(i, j, k) \mid i \leq k, j \leq k, k \leq i + j\}$ , see [3]. Hence  $h^{0,0,k}(\mathcal{M}_\mu) = 0$  if  $k > 0$ .

We thus deduce that the constant term of  $E(\mathcal{M}_\mu; q)$  is  $h^{0,0,0}(\mathcal{M}_\mu)$ .  $\square$

From the above lemma and Formula (1.1.2) we are reduced to prove that the coefficient of the lowest power  $q^{-\frac{d_\mu}{2}}$  of  $q$  in  $\mathbb{H}_\mu(\sqrt{q}, 1/\sqrt{q})$  is equal to 1.

The strategy to prove this goes in two steps. First, 5.3.1 we analyze the lowest power of  $q$  in  $\mathcal{A}_\mu(q)$ , where

$$\Omega(\sqrt{q}, 1/\sqrt{q}) = \sum_{\lambda, \mu} \mathcal{A}_\mu(q) m_\mu.$$

Then 5.3.2 we see how these combine in  $\text{Log}\left(\Omega\left(\sqrt{q}, 1/\sqrt{q}\right)\right)$ . In both case, Lemma 5.2.8 and Lemma 5.3.6, we will use in an essential way the inequality of §6. Though very similar, the relation between the partitions  $\nu^p$  in these lemmas and the matrix of numbers  $x_{i,j}$  in §6 is dual to each other (the  $\nu^p$  appear as rows in one and columns in the other).

## 5.2 Preliminaries

For a multi-partition  $\mu \in (\mathcal{P}_n)^k$  we define

$$\Delta(\mu) := \frac{1}{2}d_\mu - 1 = \frac{1}{2}(2g - 2 + k)n^2 - \frac{1}{2} \sum_{i,j} (\mu_j^i)^2. \quad (5.2.1)$$

*Remark 5.2.1.* Note that when  $g = 0$  the quantity  $-2\Delta(\mu)$  is Katz's *index of rigidity* of a solution to  $X_1 \cdots X_k = I$  with  $X_i \in C_i$  (see for example [20][p. 91]).

From  $\mu$  we define as above Theorem 3.2.3 a comet-shaped quiver  $\Gamma = \Gamma_\mu$  as well as a dimension vector  $\mathbf{v} = \mathbf{v}_\mu$  of  $\Gamma$ . We denote by  $I$  the set of vertices of  $\Gamma$  and by  $\Omega$  the set of arrows. Recall that  $\mu$  and  $\mathbf{v}$  are linearly related ( $v_0 = n$  and  $v_{[i,j]} = n - \sum_{r=1}^j \mu_r^i$  for  $j > 1$  and conversely,  $\mu_1^i = n - v_{[i,1]}$  and  $\mu_j^i = v_{[i,j-1]} - v_{[i,j]}$  for  $j > 1$ ). Hence  $\Delta$  yields an integral-valued quadratic form on  $\mathbb{Z}^I$ . Let  $(\cdot, \cdot)$  be the associated bilinear form on  $\mathbb{Z}^I$  so that

$$(\mathbf{v}, \mathbf{v}) = 2\Delta(\mu). \quad (5.2.2)$$

Let  $\mathbf{e}_0$  and  $\mathbf{e}_{[i,j]}$  be the fundamental roots of  $\Gamma$  (vectors in  $\mathbb{Z}^I$  with all zero coordinates except for a 1 at the indicated vertex). We find that

$$(\mathbf{e}_0, \mathbf{e}_0) = 2g - 2, \quad (\mathbf{e}_{[i,j]}, \mathbf{e}_{[i,j]}) = -2, \quad (\mathbf{e}_0, \mathbf{e}_{[i,1]}) = 1 \quad (\mathbf{e}_{[i,j]}, \mathbf{e}_{[i,j+1]}) = 1,$$

for  $i = 1, 2, \dots, k, j = 1, 2, \dots, s_i - 1$  and all other pairings are zero. In other words,  $\Delta$  is the negative of the Tits quadratic form of  $\Gamma$  (with the natural orientation of all edges pointing away from the central vertex).

With this notation we define

$$\delta = \delta(\mu) := (\mathbf{e}_0, \mathbf{v}) = (2g - 2 + k)n - \sum_{i=1}^k \mu_1^i. \quad (5.2.3)$$

*Remark 5.2.2.* In the case of  $g = 0$  the quantity  $\delta$  is called the *defect* by Simpson (see [29, p.12]).

Note that  $\delta \geq (2g - 2)n$  is non-negative unless  $g = 0$ . On the other hand,

$$(\mathbf{e}_{[i,j]}, \mathbf{v}) = \mu_j^i - \mu_{j+1}^i \geq 0. \quad (5.2.4)$$

We now follow the terminology of [16].

**Lemma 5.2.3.** *The dimension vector  $\mathbf{v}$  is in the fundamental set of imaginary roots of  $\Gamma$  if and only if  $\delta(\mu) \geq 0$ .*

*Proof.* Note that  $v_{[i,j]} > 0$  if  $j < l(\mu^i)$  and  $v_{[i,j]} = 0$  for  $j \geq l(\mu^i)$ ; since  $n > 0$  the support of  $\mathbf{v}$  is then connected. We already have  $(\mathbf{e}_{[i,j]}, \mathbf{v}) \geq 0$  by (5.2.4), hence  $\mathbf{v}$  is in the fundamental set of imaginary roots of  $\Gamma$  if and only if  $\delta \geq 0$  (see [16]).  $\square$

For a partition  $\mu \in \mathcal{P}_n$  we define

$$\sigma(\mu) := n\mu_1 - \sum_j \mu_j^2$$

and extend to a multipartition  $\mu \in (\mathcal{P}_n)^k$  by

$$\sigma(\mu) := \sum_{i=1}^k \sigma(\mu^i).$$

*Remark 5.2.4.* Again for  $g = 0$  this is called the *superdefect* by Simpson.

We say that  $\mu \in \mathcal{P}_n$  is *rectangular* if and only if all of its (non-zero) parts are equal, i.e.,  $\mu = (t^{n/t})$  for some  $t \mid n$ . We extend this to multi-partitions:  $\mu = (\mu^1, \dots, \mu^k) \in (\mathcal{P}_n)^k$  is rectangular if each  $\mu^i$  is (the  $\mu^i$ 's are not required to be of the same length). Note that  $\mu$  is rectangular if and only if the associated dimension vector  $\mathbf{v}$  satisfies  $(\mathbf{e}_{[i,j]}, \mathbf{v}) = 0$  for all  $[i, j]$  by (5.2.4).

**Lemma 5.2.5.** *For  $\mu \in (\mathcal{P}_n)^k$  we have*

$$\sigma(\mu) \geq 0$$

*with equality if and only if  $\mu$  is rectangular.*

*Proof.* For any  $\mu \in \mathcal{P}_n$  we have  $n\mu_1 = \mu_1 \sum_j \mu_j \geq \sum_j \mu_j^2$  and equality holds if and only if  $\mu_1 = \mu_j$ .  $\square$

Since

$$2\Delta(\mu) = n\delta(\mu) + \sigma(\mu) \quad (5.2.5)$$

we find that

$$d_\mu \geq n\delta(\mu) + 2 \quad (5.2.6)$$

and in particular  $d_\mu \geq 2$  if  $\delta(\mu) \geq 0$ .

If  $\Gamma$  is affine it is known that the positive imaginary roots are of the form  $t\mathbf{v}^*$  for an integer  $t \geq 1$  and some  $\mathbf{v}^*$ . We will call  $\mathbf{v}^*$  the *basic positive imaginary root* of  $\Gamma$ . The affine star-shaped quivers are given in the table below; their basic positive imaginary root is the dimension vector associated to the indicated multi-partition  $\mu^*$ . These  $\mu^*$ , and hence also any scaled version  $t\mu^*$  for  $t \geq 1$ , are rectangular. Moreover,  $\Delta(\mu^*) = 0$  and in fact,  $\mu^*$  generates the one-dimensional radical of the quadratic form  $\Delta$  so that  $\Delta(\mu^*, \mathbf{v}) = 0$  for all  $\mathbf{v}$ .

**Proposition 5.2.6.** *Suppose that  $\mu = (\mu^1, \dots, \mu^k) \in (\mathcal{P}_n)^k$  has  $\delta(\mu) \geq 0$ . Then  $d_\mu = 2$  if and only if  $\Gamma$  is of affine type, i.e.,  $\Gamma$  is either the Jordan quiver  $J$  (one loop on one vertex),  $\tilde{D}_4, \tilde{E}_6, \tilde{E}_7$  or  $\tilde{E}_8$ , and  $\mu = t\mu^*$  (all parts scaled by  $t$ ) for some  $t \geq 1$ , where  $\mu^*$ , given in the table below, corresponds to the basic imaginary root of  $\Gamma$ .*

*Proof.* By (5.2.5) and Lemma 5.2.5  $d_\mu = 2$  when  $\delta(\mu) \geq 0$  if and only if  $\delta(\mu) = 0$  and  $\mu$  is rectangular. As we observed above  $\delta(\mu) \geq (2g - 2)n$ . Hence if  $\delta(\mu) = 0$  then  $g = 1$  or  $g = 0$ . If  $g = 1$  then necessarily  $\mu^i = (n)$  and  $\Gamma$  is the Jordan quiver  $J$ .

If  $g = 0$  then  $\delta = 0$  is equivalent to the equation

$$\sum_{i=1}^k \frac{1}{l_i} = k - 2, \quad (5.2.7)$$

where  $l_i := n/t_i$  is the length of  $\mu^i = (t_i^{n/t_i})$ . In solving this equation, any term with  $l_i = 1$  can be ignored. It is elementary to find all of its solutions; they correspond to the cases  $\Gamma = \tilde{D}_4, \tilde{E}_6, \tilde{E}_7$  or  $\tilde{E}_8$ .

We summarize the results in the following table

$\Gamma$	$l_i$	$n$	$\mu^*$
$J$	(1)	1	(1)
$\tilde{D}_4$	(2, 2, 2, 2)	2	(1, 1), (1, 1), (1, 1), (1, 1)
$\tilde{E}_6$	(3, 3, 3)	3	(1, 1, 1), (1, 1, 1), (1, 1, 1)
$\tilde{E}_7$	(2, 4, 4)	4	(2, 2), (1, 1, 1, 1), (1, 1, 1, 1)
$\tilde{E}_8$	(2, 3, 6)	6	(3, 3), (2, 2, 2), (1, 1, 1, 1, 1, 1)

(5.2.8)

where we listed the cases with smallest possible positive values of  $n$  and  $k$  and the corresponding multi-partition  $\mu^*$ .  $\square$

Proposition 5.2.6 is due to Kostov, see for example [29, p.14].

We will need the following result about  $\Delta$ .

**Proposition 5.2.7.** *Let  $\mu \in (\mathcal{P}_n)^k$  and  $\mathbf{v}^p = (v^{1,p}, \dots, v^{k,p}) \in (\mathcal{P}_{n_p})^k$  for  $p = 1, \dots, s$  be non-zero multi-partitions such that up to permutations of the parts of  $\mathbf{v}^{i,p}$  we have*

$$\mu^i = \sum_{p=1}^s \mathbf{v}^{i,p}, \quad i = 1, \dots, k.$$

Assume that  $\delta(\mu) \geq 0$ . Then

$$\sum_{p=1}^s \Delta(\mathbf{v}^p) \leq \Delta(\mu).$$

Equality holds if and only if

(i)  $s = 1$  and  $\mu = \mathbf{v}^1$ .

or

(ii)  $\Gamma$  is affine and  $\mu, \mathbf{v}^1, \dots, \mathbf{v}^s$  correspond to positive imaginary roots.

We start with the following. For partitions  $\mu, \nu$  define

$$\sigma_\mu(\nu) := \mu_1 |\nu|^2 - |\mu| \sum_i \nu_i^2.$$

Note that  $\sigma_\mu(\mu) = |\mu| \sigma(\mu)$ .

**Lemma 5.2.8.** *Let  $\nu^1, \dots, \nu^s$  and  $\mu$  be non-zero partitions such that up to permutation of the parts of each  $\nu^p$  we have  $\sum_{p=1}^s \nu^p = \mu$ . Then*

$$\sum_{p=1}^s \sigma_{\mu}(\nu^p) \leq \sigma_{\mu}(\mu).$$

*Equality holds if and only if:*

- (i)  $s = 1$  and  $\mu = \nu^1$ .
- or
- (ii)  $\nu^1, \dots, \nu^s$  and  $\mu$  all are rectangular of the same length.

*Proof.* This is just a restatement of the inequality of §6 with  $x_{i,k} = \nu_{\sigma_k(i)}^k$ , for the appropriate permutations  $\sigma_k$ , where  $1 \leq i \leq l(\mu)$ ,  $1 \leq k \leq s$ .  $\square$

**Lemma 5.2.9.** *If the partitions  $\mu, \nu$  are rectangular of the same length then*

$$\sigma_{\mu}(\nu) = 0.$$

*Proof.* Direct calculation.  $\square$

*Proof of Proposition 5.2.7.* From the definition (5.2.1) we get

$$2n\Delta(\mu) = \delta(\mu)n^2 + \sum_{i=1}^k \sigma_{\mu^i}(\mu^i)$$

and similarly

$$2n\Delta(\nu^p) = \delta(\mu)n_p^2 + \sum_{i=1}^k \sigma_{\mu^i}(\nu^{i,p}), \quad p = 1, \dots, s$$

hence

$$2n \sum_{p=1}^s \Delta(\nu^p) = \delta(\mu) \sum_{p=1}^s n_p^2 + \sum_{i=1}^k \sum_{p=1}^s \sigma_{\mu^i}(\nu^{i,p}).$$

Since  $n = \sum_{p=1}^s n_p$  and  $\delta(\mu) \geq 0$  we get from Lemma 5.2.8 that

$$\sum_{p=1}^s \Delta(\nu^p) \leq \Delta(\mu)$$

as claimed.

Clearly, equality cannot occur if  $\delta(\mu) > 0$  and  $s > 1$ . If  $\delta(\mu) = 0$  and  $s > 1$  it follows from Lemmas 5.2.8, 5.2.9 and (5.2.5) that  $\Delta(\mu) = \Delta(\nu^p) = 0$  for  $p = 1, 2, \dots, s$ . Now (ii) is a consequence of Proposition 5.2.6.  $\square$

### 5.3 Proof of Theorem 5.1.1

#### 5.3.1 Step I

Let

$$\mathcal{A}_{\lambda\mu}(q) := q^{(1-g)|\lambda|} \left( q^{-n(\lambda)} H_{\lambda}(q) \right)^{2g+k-2} \prod_{i=1}^k \langle h_{\mu^i}(\mathbf{x}_i), s_{\lambda}(\mathbf{x}_i \mathbf{y}) \rangle, \quad (5.3.1)$$

so that by Lemma 2.1.5

$$\Omega(\sqrt{q}, 1/\sqrt{q}) = \sum_{\lambda, \mu} \mathcal{A}_{\lambda\mu}(q) m_{\mu}.$$

It is easy to verify that  $\mathcal{A}_{\lambda\mu}$  is in  $\mathbb{Q}(q)$ .



For a non-zero rational function  $\mathcal{A} \in \mathbb{Q}(q)$  we let  $v_q(\mathcal{A}) \in \mathbb{Z}$  be its valuation at  $q$ . We will see shortly that  $\mathcal{A}_{\lambda\mu}$  is non-zero for all  $\lambda, \mu$ ; let  $v(\lambda) := v_q(\mathcal{A}_{\lambda\mu}(q))$ . The first main step toward the proof of the connectedness is the following theorem.

**Theorem 5.3.1.** *Let  $\mu = (\mu^1, \mu^2, \dots, \mu^k) \in \mathcal{P}_n^k$  with  $\delta(\mu) \geq 0$ . Then*

*i) The minimum value of  $v(\lambda)$  as  $\lambda$  runs over the set of partitions of size  $n$ , is*

$$v((1^n)) = -\Delta(\mu).$$

*ii) There are two cases as to where this minimum occurs.*

*Case I: The quiver  $\Gamma$  is affine and the dimension vector associated to  $\mu$  is a positive imaginary root  $t\mathbf{v}^*$  for some  $t \mid n$ . In this case, the minimum is reached at all partitions  $\lambda$  which are the union of  $n/t$  copies of any  $\lambda_0 \in \mathcal{P}_t$ .*

*Case II: Otherwise, the minimum occurs only at  $\lambda = (1^n)$ .*

Before proving the theorem we need some preliminary results.

**Lemma 5.3.2.**  $\langle h_\mu(\mathbf{x}), s_\lambda(\mathbf{xy}) \rangle$  is non-zero for all  $\lambda$  and  $\mu$ .

*Proof.* We have  $s_\lambda(\mathbf{xy}) = \sum_v K_{\lambda v} m_v(\mathbf{xy})$  [24, I 6 p.101] and  $m_v(\mathbf{xy}) = \sum_\mu C_{v\mu}(\mathbf{y}) m_\mu(\mathbf{x})$  for some  $C_{v\mu}(\mathbf{y})$ . Hence

$$\langle h_\mu(\mathbf{x}), s_\lambda(\mathbf{xy}) \rangle = \sum_v K_{\lambda v} C_{v\mu}(\mathbf{y}). \quad (5.3.2)$$

For any set of variables  $\mathbf{xy} = \{x_i y_j\}_{1 \leq i, 1 \leq j}$  we have

$$C_{v\mu}(\mathbf{y}) = \sum m_{\rho^1}(\mathbf{y}) \cdots m_{\rho^r}(\mathbf{y}), \quad (5.3.3)$$

where the sum is over all partitions  $\rho^1, \dots, \rho^r$  such that  $|\rho^p| = \mu_p$  and  $\rho^1 \cup \dots \cup \rho^r = v$ . In particular the coefficients of  $C_{v\mu}(\mathbf{y})$  as power series in  $q$  are non-negative. We can take, for example,  $\rho^p = (1^{\mu_p})$  and then  $v = (1^n)$ . Since  $K_{\lambda v} \geq 0$  [24, I (6.4)] for any  $\lambda, v$  and  $K_{\lambda, (1^n)} = n!/h_\lambda$  [24, I 6 ex. 2], with  $h_\lambda = \prod_{s \in \lambda} h(s)$  the product of the hook lengths, we see that  $\langle h_\mu(\mathbf{x}), s_\lambda(\mathbf{xy}) \rangle$  is non-zero and our claim follows.  $\square$

In particular  $\mathcal{A}_{\lambda\mu}$  is non-zero for all  $\lambda$  and  $\mu$ . Define

$$v(\lambda, \mu) := v_q(\langle h_\mu(\mathbf{x}), s_\lambda(\mathbf{xy}) \rangle). \quad (5.3.4)$$

**Lemma 5.3.3.** *We have*

$$-v(\lambda) = (2g - 2 + k)n(\lambda) + (g - 1)n - \sum_{i=1}^k v(\lambda, \mu^i).$$

*Proof.* Straightforward.  $\square$

**Lemma 5.3.4.** *For  $\mu = (\mu_1, \mu_2, \dots, \mu_r) \in \mathcal{P}_n$  we have*

$$v(\lambda, \mu) = \min\{n(\rho^1) + \dots + n(\rho^r) \mid |\rho^p| = \mu_p, \cup_p \rho^p \trianglelefteq \lambda\}. \quad (5.3.5)$$

*Proof.* For  $C_{v\mu}(\mathbf{y})$  non-zero let  $v_m(v, \mu) := v_q(C_{v\mu}(\mathbf{y}))$ . When  $y_i = q^{i-1}$  we have  $v_q(m_\rho(\mathbf{y})) = n(\rho)$  for any partition  $\rho$ . Hence by (5.3.3)

$$v_m(v, \mu) = \min\{n(\rho^1) + \dots + n(\rho^r) \mid |\rho^p| = \mu_p, \cup_p \rho^p = v\}.$$

Since  $K_{\lambda v} \geq 0$  for any  $\lambda, v$ ,  $K_{\lambda v} > 0$  if and only if  $v \leq \lambda$  [6, Ex 2, p.26], and the coefficients of  $C_{v\mu}(\mathbf{y})$  are non-negative, our claim follows from (5.3.2).  $\square$

For example, if  $\lambda = (1^n)$  then necessarily  $\rho^p = (1^{\mu_p})$  and hence  $\rho^1 \cup \dots \cup \rho^r = \lambda$ . We have then

$$v((1^n), \mu) = \sum_{p=1}^r \binom{\mu_p}{2} = -\frac{1}{2}n + \frac{1}{2} \sum_{p=1}^r \mu_p^2. \quad (5.3.6)$$

Similarly,

$$v(\lambda, (n)) = n(\lambda) \quad (5.3.7)$$

by the next lemma.

**Lemma 5.3.5.** *If  $\beta \trianglelefteq \alpha$  then  $n(\alpha) \leq n(\beta)$  with equality if and only if  $\alpha = \beta$ .*

*Proof.* We will use the raising operators  $R_{ij}$  see [24, I p.8]. Consider vectors  $w$  with coefficients in  $\mathbb{Z}$  and extend the function  $n$  to them in the natural way

$$n(w) := \sum_{i \geq 1} (i-1)w_i.$$

Applying a raising operator  $R_{ij}$ , where  $i < j$ , has the effect

$$n(R_{ij}w) = n(w) + i - j.$$

Hence for any product  $R$  of raising operators we have  $n(Rw) < n(w)$  with equality if and only if  $R$  is the identity operator. Now the claim follows from the fact that  $\beta \trianglelefteq \alpha$  implies there exist such  $R$  with  $\alpha = R\beta$ .  $\square$

Recall [24, (1.6)] that for any partition  $\lambda$  we have  $\langle \lambda, \lambda \rangle = 2n(\lambda) + |\lambda| = \sum_i (\lambda'_i)^2$ , where  $\lambda' = (\lambda'_1, \lambda'_2, \dots)$  is the dual partition. Note also that  $(\lambda \cup \mu)' = \lambda' + \mu'$ . Define

$$\|\lambda\| := \sqrt{\langle \lambda', \lambda' \rangle} = \sqrt{\sum_i \lambda_i^2}.$$

The following inequality is a particular case of the theorem of §6.

**Lemma 5.3.6.** *Fix  $\mu = (\mu_1, \mu_2, \dots, \mu_r) \in \mathcal{P}_n$ . Then for every  $(v^1, \dots, v^r) \in \mathcal{P}_{\mu_1} \times \dots \times \mathcal{P}_{\mu_r}$  we have*

$$\mu_1 \left\| \sum_p v^p \right\|^2 - n \sum_p \|v^p\|^2 \leq \mu_1 n^2 - n \|\mu\|^2. \quad (5.3.8)$$

Moreover, equality holds in (5.3.8) if and only if either:

- (i) The partition  $\mu$  is rectangular and all partitions  $v^p$  are equal.
- or
- (ii) For each  $p = 1, 2, \dots, r$  we have  $v^p = (\mu_p)$ .

*Proof.* Our claim is a consequence of the theorem of §6. Taking  $x_{ps} = v_s^p$  we have  $c_p := \sum_s x_{ps} = \sum_s v_s^p = \mu_p$  and  $c := \max_p c_p = \mu_1$ .  $\square$

The following fact will be crucial for the proof of connectedness.

**Proposition 5.3.7.** *For a fixed  $\mu = (\mu_1, \mu_2, \dots, \mu_r) \in \mathcal{P}_n$  we have*

$$\mu_1 n(\lambda) - n v(\lambda, \mu) \leq \mu_1 n^2 - n \|\mu\|^2, \quad \lambda \in \mathcal{P}_n.$$

Equality holds only at  $\lambda = (1^n)$  unless  $\mu$  is rectangular  $\mu = (t^{n/t})$ , in which case it also holds when  $\lambda$  is the union of  $n/t$  copies of any  $\lambda_0 \in \mathcal{P}_t$ .

*Proof.* Given  $\nu \trianglelefteq \lambda$  write  $\mu_1 n(\lambda) - n\nu(\lambda, \mu)$  as

$$\mu_1 n(\lambda) - n\nu(\lambda, \mu) = \mu_1 (n(\lambda) - n(\nu)) + \mu_1 n(\nu) - n\nu(\lambda, \mu) \quad (5.3.9)$$

By Lemma 5.3.5 the first term is non-negative. Hence

$$\mu_1 n(\lambda) - n\nu(\lambda, \mu) \leq \mu_1 n(\nu) - n\nu(\lambda, \mu), \quad \nu \trianglelefteq \lambda.$$

Combining this with (5.3.5) yields

$$\max_{|\lambda|=n} [\mu_1 n(\lambda) - n\nu(\lambda, \mu)] \leq \max_{|\rho^p|=\mu_p} [\mu_1 n(\rho^1 \cup \rho^2 \cup \dots \cup \rho^r) - (n(\rho^1) + \dots + n(\rho^r))n]. \quad (5.3.10)$$

Take  $\nu^p$  to be the dual of  $\rho^p$  for  $p = 1, 2, \dots, r$ . Then the right hand side of (5.3.10) is precisely

$$\mu_1 \left\| \sum_p \nu^p \right\|^2 - n \sum_p \|\nu^p\|^2,$$

which by Lemma 5.3.6 is bounded above by  $\mu_1 n^2 - n \|\mu\|^2$  with equality only where either  $\rho^p = (1^{\mu_p})$  (case (ii)) or all  $\rho^p$  are equal and  $\mu = (t^{n/t})$  for some  $t$  (case (i)).

Combining this with Lemma 5.3.5 we see that to obtain the maximum of the left hand side of (5.3.10) we must also have  $\rho^1 \cup \dots \cup \rho^r = \lambda$ . In case (i) then,  $\lambda$  is the union of  $n/t$  copies of  $\lambda_0$ , the common value of  $\rho^p$ , and in case (ii),  $\lambda = (1^n)$ .  $\square$

*Proof of Theorem 5.3.1.* We first prove (ii). Using Lemma 5.3.3 we have

$$-v(\lambda) = (2g - 2 + k)n(\lambda) + (g - 1)n - \sum_{i=1}^k v(\lambda, \mu^i) = \frac{\delta}{n}n(\lambda) + (g - 1)n + \frac{1}{n} \sum_{i=1}^k [\mu_1^i n(\lambda) - n\nu(\lambda, \mu^i)]. \quad (5.3.11)$$

The terms  $n(\lambda)$  and  $\sum_{i=1}^k [\mu_1^i n(\lambda) - n\nu(\lambda, \mu^i)]$  are all maximal at  $\lambda = (1^n)$  (the last by Proposition 5.3.7). Hence  $-v(\lambda)$  is also maximal at  $(1^n)$ , since  $\delta \geq 0$ . Now  $n(\lambda)$  has a unique maximum at  $(1^n)$  by Lemma 5.3.5, hence  $-v(\lambda)$  reaches its maximum at other partitions if and only if  $\delta = 0$  and for each  $i$  we have  $\mu^i = (t_i^{n/t_i})$  for some positive integer  $t_i \mid n$  (again by Proposition 5.3.7). In this case the maximum occurs only for  $\lambda$  the union of  $n/t$  copies of a partition  $\lambda_0 \in \mathcal{P}_t$ , where  $t = \gcd t_i$ . Now (ii) follows from Proposition 5.2.6.

To prove (i) we use Lemma 5.3.3 and (5.3.6) and find that  $v((1^n)) = -\Delta(\mu)$  as claimed.  $\square$

**Lemma 5.3.8.** *Let  $\mu = (\mu^1, \mu^2, \dots, \mu^k) \in \mathcal{P}_n^k$  with  $\delta(\mu) \geq 0$ . Suppose that  $v(\lambda)$  is minimal. Then the coefficient of  $q^{v(\lambda)}$  in  $\mathcal{A}_{\lambda\mu}$  is 1.*

*Proof.* We use the notation of the proof of Lemma 5.3.4. Note that the coefficient of the lowest power of  $q$  in  $\mathcal{H}_\lambda(\sqrt{q}, 1/\sqrt{q})(q^{-n(\lambda)}H_\lambda(q))^k$  is 1 (see (2.1.10)). Also, the coefficient of the lowest power of  $q$  in each  $m_\lambda(\mathbf{y})$  is always 1; hence so is the coefficient of the lowest power of  $q$  in  $C_{\nu\mu}(\mathbf{y})$ .

In the course of the proof of Proposition 5.3.7 we found that when  $v(\lambda)$  is minimal, and  $\rho^1, \dots, \rho^r$  achieve the minimum in the right hand side of (5.3.5), then  $\lambda = \rho^1 \cup \dots \cup \rho^r$ . Hence by Lemma 5.3.4, the coefficient of the lowest power of  $q$  in  $\langle h_\mu(\mathbf{x}), s_\lambda(\mathbf{xy}) \rangle = \sum_{\nu \trianglelefteq \lambda} K_{\lambda\nu} C_{\nu\mu}(\mathbf{y})$  equals the coefficient of the lowest power of  $q$  in  $K_{\lambda\lambda} C_{\lambda\mu}(\mathbf{y}) = C_{\lambda\mu}(\mathbf{y})$  which we just saw is 1. This completes the proof.  $\square$

### 5.3.2 Leading terms of $\text{Log } \Omega$

We now proceed to the second step in the proof of connectedness where we analyze the smallest power of  $q$  in the coefficients of  $\text{Log}(\Omega(\sqrt{q}, 1/\sqrt{q}))$ . Write

$$\Omega(\sqrt{q}, 1/\sqrt{q}) = \sum_{\mu} P_{\mu}(q) m_{\mu} \quad (5.3.12)$$

with  $P_\mu(q) := \sum_\lambda \mathcal{A}_{\lambda\mu}$  and  $\mathcal{A}_{\lambda\mu}$  as in (5.3.1).

Then by Lemma 2.1.4 we have

$$\text{Log}(\Omega(\sqrt{q}, 1/\sqrt{q})) = \sum_{\omega} C_{\omega}^0 P_{\omega}(q) m_{\omega}(q)$$

where  $\omega$  runs over *multi-types*  $(d_1, \omega^1) \cdots (d_s, \omega^s)$  with  $\omega^p \in (\mathcal{P}_{n_p})^k$  and  $P_{\omega}(q) := \prod_p P_{\omega^p}(q^{d_p})$ ,  $m_{\omega}(\mathbf{x}) := \prod_p m_{\omega^p}(\mathbf{x}^{d_p})$ .

Now if we let  $\gamma_{\mu\omega} := \langle m_{\omega}, h_{\mu} \rangle$  then we have

$$\mathbb{H}_{\mu}(\sqrt{q}, 1/\sqrt{q}) = \frac{(q-1)^2}{q} \left( \sum_{\omega \in \mathbf{T}^k} C_{\omega}^0 P_{\omega}(q) \gamma_{\mu\omega} \right).$$

By Theorem 5.3.1,  $v_q(P_{\omega}(q)) = -d \sum_{p=1}^s \Delta(\omega^p)$  for a multi-type  $\omega = (d, \omega^1) \cdots (d, \omega^s)$ .

**Lemma 5.3.9.** *Let  $v^1, \dots, v^s$  be partitions. Then*

$$\langle m_{v^1} \cdots m_{v^s}, h_{\mu} \rangle \neq 0$$

*if and only if  $\mu = v^1 + \cdots + v^s$  up to permutation of the parts of each  $v^p$  for  $p = 1, \dots, s$ .*

*Proof.* It follows immediately from the definition of the monomial symmetric function.  $\square$

Let  $\mathbf{v}$  be the dimension vector associated to  $\mu$ .

**Theorem 5.3.10.** *If  $\mathbf{v}$  is in the fundamental set of imaginary roots of  $\Gamma$  then the character variety  $\mathcal{M}_{\mu}$  is non-empty and connected.*

*Proof.* Assume  $\mathbf{v}$  is in the fundamental set of roots of  $\Gamma$ . By Lemma 5.2.3 this is equivalent to  $\delta(\mu) \geq 0$ .

Note that  $m_v(\mathbf{x}^d) = m_{dv}(\mathbf{x})$  for any partition  $v$  and positive integer  $d$ . Suppose  $\omega = (d, \omega^1) \cdots (d, \omega^s)$  is a multi-type for which  $\gamma_{\mu\omega}$  is non-zero. Let  $v^p = d\omega^p$  for  $p = 1, \dots, s$  (scale every part by  $d$ ). These multi-partitions are then exactly in the hypothesis of Proposition 5.2.7 by Lemma 5.3.9. Hence

$$d \sum_{p=1}^s \Delta(\omega^p) \leq d^2 \sum_{p=1}^s \Delta(\omega^p) = \sum_{p=1}^s \Delta(v^p) \leq \Delta(\mu). \quad (5.3.13)$$

Suppose  $\Gamma$  is not affine. Then by Proposition 5.2.7 we have equality of the endpoints in (5.3.13) if and only if  $s = 1$ ,  $v^1 = \mu$  and  $d = 1$ , in other words, if and only if  $\omega = (1, \mu)$ . Hence, since  $C_{(1, \mu)}^0 = 1$ , the coefficient of the lowest power of  $q$  in  $\mathbb{H}_{\mu}(\sqrt{q}, 1/\sqrt{q})$  equals the coefficient of the lowest power of  $q$  in  $P_{\mu}(q)$  which is 1 by Lemma 5.3.8 and Theorem 5.3.1, Case II. This proves our claim in this case.

Suppose now  $\Gamma$  is affine. Then by Proposition 5.2.7 we have equality of the endpoints in (5.3.13) if and only if  $\mu = t\mu^*$  and  $\omega = (1, t_1\mu^*), \dots, (1, t_s\mu^*)$  for a partition  $(t_1, t_2, \dots, t_s)$  of  $t$  and  $d = 1$ . Combining this with Lemma 5.3.8 and Theorem 5.3.1, Case I we see that the lowest order terms in  $q$  in  $\text{Log}(\Omega(\sqrt{q}, 1/\sqrt{q}))$  are

$$L := \sum C_{\omega}^0 p(t_1) \cdots p(t_s) m_{\mu^*},$$

where the sum is over types  $\omega$  as above. Comparison with Euler's formula

$$\text{Log} \left( \sum_{n \geq 0} p(n) T^n \right) = \sum_{n \geq 1} T^n,$$

shows that  $L$  reduces to  $\sum_{t \geq 1} m_{\mu^*}$ . Hence the coefficient of the lowest power of  $q$  in  $\mathbb{H}_{\mu}(\sqrt{q}, 1/\sqrt{q})$  is also 1 in this case finishing the proof.  $\square$

*Proof of Theorem 5.1.1.* If  $g \geq 1$ , the dimension vector  $\mathbf{v}$  is always in the fundamental set of imaginary roots of  $\Gamma$ . If  $g = 0$  the character variety is not empty if and only if  $\mathbf{v}$  is a strict root of  $\Gamma$  and if  $\mathbf{v}$  is real then  $\mathcal{M}_\mu$  is a point [2, Theorem 8.3]. If  $\mathbf{v}$  is imaginary then it can be taken by the Weyl group to some  $\mathbf{v}'$  in the fundamental set and the two corresponding varieties  $\mathcal{M}_\mu$  and  $\mathcal{M}_{\mu'}$  are isomorphic for appropriate choices of conjugacy classes [2, Theorem 3.2, Lemma 4.3 (ii)], hence Theorem 5.1.1.  $\square$

## 6 Appendix by Gergely Harcos

**Theorem 6.0.11.** *Let  $n, r$  be positive integers, and let  $x_{ik}$  ( $1 \leq i \leq n$ ,  $1 \leq k \leq r$ ) be arbitrary nonnegative numbers. Let  $c_i := \sum_k x_{ik}$  and  $c := \max_i c_i$ . Then we have*

$$c \sum_k \left( \sum_i x_{ik} \right)^2 - \left( \sum_i c_i \right) \left( \sum_{i,k} x_{ik}^2 \right) \leq c \left( \sum_i c_i \right)^2 - \left( \sum_i c_i \right) \left( \sum_i c_i^2 \right).$$

Assuming  $\min_i c_i > 0$ , equality holds if and only if we are in one of the following situations

- (i)  $x_{ik} = x_{jk}$  for all  $i, j, k$ ,
- (ii) there exists some  $l$  such that  $x_{ik} = 0$  for all  $i$  and all  $k \neq l$ .

*Remark 6.0.12.* The assumption  $\min_i c_i > 0$  does not result in any loss of generality, because the values  $i$  with  $c_i = 0$  can be omitted without altering any of the sums.

*Proof.* Without loss of generality we can assume  $c = c_1 \geq \dots \geq c_n$ , then the inequality can be rewritten as

$$\left( \sum_i c_i \right) \left( \sum_j \sum_{k,l} x_{jk} x_{jl} - \sum_{j,k} x_{jk}^2 \right) \leq c \left( \sum_{i,j} \sum_{k,l} x_{ik} x_{jl} - \sum_{i,j} \sum_k x_{ik} x_{jk} \right).$$

Here and later  $i, j$  will take values from  $\{1, \dots, n\}$  and  $k, l, m$  will take values from  $\{1, \dots, r\}$ . We simplify the above as

$$\left( \sum_i c_i \right) \left( \sum_j \sum_{\substack{k,l \\ k \neq l}} x_{jk} x_{jl} \right) \leq c \left( \sum_{i,j} \sum_{\substack{k,l \\ k \neq l}} x_{ik} x_{jl} \right),$$

then we factor out and also utilize the symmetry in  $k, l$  to arrive at the equivalent form

$$\sum_{i,j} c_i \sum_{\substack{k,l \\ k < l}} x_{jk} x_{jl} \leq \sum_{i,j} c \sum_{\substack{k,l \\ k < l}} x_{ik} x_{jl}.$$

We distribute the terms in  $i, j$  on both sides as follows:

$$\sum_i c_i \sum_{\substack{k,l \\ k < l}} x_{ik} x_{il} + \sum_{\substack{i,j \\ i < j}} \left( c_i \sum_{\substack{k,l \\ k < l}} x_{jk} x_{jl} + c_j \sum_{\substack{k,l \\ k < l}} x_{ik} x_{il} \right) \leq \sum_i c \sum_{\substack{k,l \\ k < l}} x_{ik} x_{il} + \sum_{\substack{i,j \\ i < j}} c \sum_{\substack{k,l \\ k < l}} (x_{ik} x_{jl} + x_{jk} x_{il}).$$

It is clear that

$$c_i \sum_{\substack{k,l \\ k < l}} x_{ik} x_{il} \leq c \sum_{\substack{k,l \\ k < l}} x_{ik} x_{il}, \quad 1 \leq i \leq n,$$

therefore it suffices to show that

$$c_i \sum_{\substack{k,l \\ k < l}} x_{jk} x_{jl} + c_j \sum_{\substack{k,l \\ k < l}} x_{ik} x_{il} \leq c \sum_{\substack{k,l \\ k < l}} (x_{ik} x_{jl} + x_{jk} x_{il}), \quad 1 \leq i < j \leq n.$$

We will prove this in the stronger form

$$c_i \sum_{\substack{k,l \\ k < l}} x_{jk} x_{jl} + c_j \sum_{\substack{k,l \\ k < l}} x_{ik} x_{il} \leq c_i \sum_{\substack{k,l \\ k < l}} (x_{ik} x_{jl} + x_{jk} x_{il}), \quad 1 \leq i < j \leq n.$$

We now fix  $1 \leq i < j \leq n$  and introduce  $x_k := x_{ik}$ ,  $x'_k := x_{jk}$ . Then the previous inequality reads

$$\left(\sum_m x_m\right)\left(\sum_{\substack{k,l \\ k < l}} x'_k x'_l\right) + \left(\sum_m x'_m\right)\left(\sum_{\substack{k,l \\ k < l}} x_k x_l\right) \leq \left(\sum_m x_m\right)\sum_{\substack{k,l \\ k < l}} (x_k x'_l + x'_k x_l),$$

that is,

$$\sum_{\substack{k,l,m \\ k < l}} (x_m x'_k x'_l + x_k x_l x'_m) \leq \sum_{\substack{k,l,m \\ k < l}} (x_k x_m x'_l + x_l x_m x'_k).$$

The right hand side equals

$$\begin{aligned} \sum_{\substack{k,l,m \\ k < l}} (x_k x_m x'_l + x_l x_m x'_k) &= \sum_{\substack{k,l,m \\ l \neq k}} x_k x_m x'_l = \sum_{\substack{k,l,m \\ m \neq k}} x_k x_l x'_m = \sum_{\substack{k,m \\ m \neq k}} x_k^2 x'_m + \sum_{\substack{k,l,m \\ l \neq k \\ m \neq k}} x_k x_l x'_m \\ &= \sum_{\substack{k,m \\ m \neq k}} x_k^2 x'_m + \sum_{\substack{k,m \\ m \neq k}} x_k x_m x'_m + \sum_{\substack{k,l,m \\ l \neq k \\ m \neq k,l}} x_k x_l x'_m \\ &= \sum_{\substack{k,m \\ m \neq k}} x_k^2 x'_m + \sum_{\substack{k,m \\ k < m}} x_k x_m x'_m + \sum_{\substack{k,m \\ m < k}} x_k x_m x'_m + 2 \sum_{\substack{k,l,m \\ k < l \\ m \neq k,l}} x_k x_l x'_m \\ &= \sum_{\substack{k,m \\ m \neq k}} x_k^2 x'_m + \sum_{\substack{k,l \\ k < l}} x_k x_l x'_l + \sum_{\substack{k,l \\ k < l}} x_k x_l x'_k + 2 \sum_{\substack{k,l,m \\ k < l \\ m \neq k,l}} x_k x_l x'_m \\ &= \sum_{\substack{k,m \\ m \neq k}} x_k^2 x'_m + \sum_{\substack{k,l,m \\ k < l}} x_k x_l x'_m + \sum_{\substack{k,l,m \\ k < l \\ m \neq k,l}} x_k x_l x'_m, \end{aligned}$$

therefore it suffices to prove

$$\sum_{\substack{k,l,m \\ k < l}} x_m x'_k x'_l \leq \sum_{\substack{k,m \\ m \neq k}} x_k^2 x'_m + \sum_{\substack{k,l,m \\ k < l \\ m \neq k,l}} x_k x_l x'_m.$$

This is trivial if  $x'_m = 0$  for all  $m$ . Otherwise  $\sum_m x'_m > 0$ , hence  $c_i \geq c_j$  yields

$$\lambda := \left(\sum_m x_m\right)\left(\sum_m x'_m\right)^{-1} \geq 1.$$

Clearly, we are done if we can prove

$$\lambda^2 \sum_{\substack{k,l,m \\ k < l}} x_m x'_k x'_l \leq \lambda \sum_{\substack{k,m \\ m \neq k}} x_k^2 x'_m + \lambda \sum_{\substack{k,l,m \\ k < l \\ m \neq k,l}} x_k x_l x'_m.$$

We introduce  $\tilde{x}_m := \lambda x'_m$ , then

$$\sum_m \tilde{x}_m = \sum_m x_m,$$

and the last inequality reads

$$\sum_{\substack{k,l,m \\ k < l}} x_m \tilde{x}_k \tilde{x}_l \leq \sum_{\substack{k,m \\ m \neq k}} x_k^2 \tilde{x}_m + \sum_{\substack{k,l,m \\ k < l \\ m \neq k,l}} x_k x_l \tilde{x}_m.$$

By adding equal sums to both sides this becomes

$$\sum_{\substack{k,l,m \\ k < l}} x_m \tilde{x}_k \tilde{x}_l + \sum_{\substack{k,l,m \\ k < l}} x_k x_l \tilde{x}_m \leq \sum_{\substack{k,m \\ m \neq k}} x_k^2 \tilde{x}_m + \sum_{\substack{k,l,m \\ k < l \\ m \neq k,l}} x_k x_l \tilde{x}_m + \sum_{\substack{k,l,m \\ k < l}} x_k x_l \tilde{x}_m,$$

which can also be written as

$$\left(\sum_m x_m\right)\left(\sum_{\substack{k,l \\ k < l}} \tilde{x}_k \tilde{x}_l\right) + \left(\sum_m \tilde{x}_m\right)\left(\sum_{\substack{k,l \\ k < l}} x_k x_l\right) \leq \sum_k x_k^2 \left(\sum_{\substack{m \\ m \neq k}} \tilde{x}_m\right) + \sum_{\substack{k,l \\ k < l}} x_k x_l \left(\sum_{\substack{m \\ m \neq k}} \tilde{x}_m + \sum_{\substack{m \\ m \neq l}} \tilde{x}_m\right).$$

The right hand side equals

$$\begin{aligned} \sum_k x_k^2 \left(\sum_{\substack{m \\ m \neq k}} \tilde{x}_m\right) + \sum_{\substack{k,l \\ k < l}} x_k x_l \left(\sum_{\substack{m \\ m \neq k}} \tilde{x}_m + \sum_{\substack{m \\ m \neq l}} \tilde{x}_m\right) &= \sum_k x_k^2 \left(\sum_{\substack{m \\ m \neq k}} \tilde{x}_m\right) + \sum_{\substack{k,l \\ l < k}} x_k x_l \left(\sum_{\substack{m \\ m \neq l}} \tilde{x}_m\right) + \sum_{\substack{k,l \\ k < l}} x_k x_l \left(\sum_{\substack{m \\ m \neq l}} \tilde{x}_m\right) \\ &= \sum_k x_k^2 \left(\sum_{\substack{m \\ m \neq k}} \tilde{x}_m\right) + \sum_{\substack{k,l \\ k \neq l}} x_k x_l \left(\sum_{\substack{m \\ m \neq l}} \tilde{x}_m\right) \\ &= \sum_{\substack{k,l \\ m \neq l}} x_k x_l \left(\sum_{\substack{m \\ m \neq l}} \tilde{x}_m\right) = \left(\sum_k x_k\right) \left(\sum_{\substack{m,l \\ m \neq l}} x_l \tilde{x}_m\right), \end{aligned}$$

hence the previous inequality is the same as

$$\left(\sum_m x_m\right)\left(\sum_{\substack{k,l \\ k < l}} \tilde{x}_k \tilde{x}_l\right) + \left(\sum_m \tilde{x}_m\right)\left(\sum_{\substack{k,l \\ k < l}} x_k x_l\right) \leq \left(\sum_k x_k\right) \left(\sum_{\substack{m,l \\ m \neq l}} x_l \tilde{x}_m\right).$$

The first factors are equal and positive, hence after renaming  $m, l$  to  $k, l$  when  $m < l$  and to  $l, k$  when  $m > l$  on the right hand side we are left with proving

$$\sum_{\substack{k,l \\ k < l}} (\tilde{x}_k \tilde{x}_l + x_k x_l) \leq \sum_{\substack{k,l \\ k < l}} (\tilde{x}_k x_l + x_k \tilde{x}_l).$$

This can be written in the elegant form

$$\sum_{\substack{k,l \\ k < l}} (\tilde{x}_k - x_k)(\tilde{x}_l - x_l) \leq 0.$$

However,

$$0 = \left(\sum_k (\tilde{x}_k - x_k)\right)^2 = \sum_{k,l} (\tilde{x}_k - x_k)(\tilde{x}_l - x_l) = \sum_k (\tilde{x}_k - x_k)^2 + 2 \sum_{\substack{k,l \\ k < l}} (\tilde{x}_k - x_k)(\tilde{x}_l - x_l),$$

so that

$$\sum_{\substack{k,l \\ k < l}} (\tilde{x}_k - x_k)(\tilde{x}_l - x_l) = -\frac{1}{2} \sum_k (\tilde{x}_k - x_k)^2 \leq 0$$

as required.

We now verify, under the assumption  $\min_i c_i > 0$ , that equation in the theorem holds if and only if  $x_{ik} = x_{jk}$  for all  $i, j, k$  or there exists some  $l$  such that  $x_{ik} = 0$  for all  $i$  and all  $k \neq l$ . The “if” part is easy, so we focus on the “only if” part. Inspecting the above argument carefully, we can see that equation can hold only if for any  $1 \leq i < j \leq n$  the numbers  $x_k := x_{ik}, x'_k := x_{jk}$  satisfy

$$\lambda \sum_{\substack{k,l,m \\ k < l}} x_m x'_k x'_l = \sum_{\substack{k,l,m \\ k < l}} x_m x'_k x'_l = \sum_{\substack{k,m \\ m \neq k}} x_k^2 x'_m + \sum_{\substack{k,l,m \\ k < l \\ m \neq k, l}} x_k x_l x'_m,$$

where  $\lambda$  is as before. If  $x'_k x'_l = 0$  for all  $k < l$ , then  $x_k^2 x'_m = 0$  for all  $k \neq m$ , i.e.  $x_k x'_l = 0$  for all  $k \neq l$ . Otherwise  $\lambda = 1$  and  $x_k = \tilde{x}_k = x'_k$  for all  $k$  by the above argument. In other words, equation in the theorem can hold only if for any  $i \neq j$  we have  $x_{ik} x_{jl} = 0$  for all  $k \neq l$  or we have  $x_{ik} = x_{jk}$  for all  $k$ . If there exist  $j, l$  such that  $x_{jk} = 0$  for all  $k \neq l$ , then  $x_{jl} > 0$  and for any  $i \neq j$  both alternatives imply  $x_{ik} = 0$  for all  $k \neq l$ , hence we are done. Otherwise the first alternative cannot hold for any  $i \neq j$ , so we are again done.  $\square$

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